CHERN APPROXIMATIONS FOR GENERALISED GROUP COHOMOLOGY

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Let G be a finite group, and let E^* be a generalised cohomology theory, subject to certain technical conditions ("admissibility" in the sense of [4]). Our aim in this paper is to define and study a certain ring $C(E,G)$ that is in a precise sense the best possible approximation to E^0BG that can be built using only knowledge of the complex representation theory of G. There is a natural map $C(E,G) \to E^0BG$, whose image is the subring $\overline{C}(E,G) \leq E^0BG$ generated over E^0 by all Chern classes of all complex representations. There is ample precedent for considering this subring in the parallel case of ordinary cohomology; see for example [13, 14, 3]. However, although the generators of $\overline{C}(E,G)$ come from representation theory, the same cannot be said for the relations; one purpose of our construction is to remedy this. We also also develop a kind of generalised character theory which gives good information about $\mathbb{Q}\otimes C(E,G)$. In the few cases that we have been able to analyse completely, either $\mathbb{Q}\otimes C(E,G) \neq \mathbb{Q}\otimes E^0BG$ for easy character-theoretic reasons, or we have $C(E,G) = E^0BG$.

Rather than working directly with rings, we will study the formal schemes $X(G) = \operatorname{spf}(E^0BG)$ and $X_{\mathrm{Ch}}(G) = \operatorname{spf}(C(E,G))$; note that the map $C(E,G) \to E^0BG$ corresponds to a map $X(G) \to X_{\mathrm{Ch}}(G)$. See [4, 9, 8] for foundational material on formal schemes. Suitably interpreted, our main definition is that $X_{\mathrm{Ch}}(G)$ is the scheme of homomorphisms from the Λ-semiring $R^+(G)$ of complex representations of G to the Λ-semiring scheme of divisors on the formal group $\mathbb{G}$ associated to E.

We start by fixing some conventions in Section 1. We then recall the basic theory of Λ-semirings (Section 2), set up the parallel theory of Λ-semiring schemes, and define the Λ-semiring scheme of divisors (Section 3). We then recall the definition of Adams operations and study their basic properties (Section 4). Using this we give a precise definition of $X_{\mathrm{Ch}}(G)$ and an implicit presentation of $C(E,G)$ by generators and relations (Section 5). In Section 6, we work out the case of the symmetric group Σ_3 at the prime 3, and show that $X(\Sigma_3) = X_{\mathrm{Ch}}(\Sigma_3)$. In Section 7 we show that $X(G) = X_{\mathrm{Ch}}(G)$ when G is Abelian, and in Section 8 we show that the same is true when E is the p-adic completion of complex K-theory and G is a p-group. We then use Adams operations to reduce certain questions to the Sylow subgroup of G (Section 9) and to prove that $X_{\mathrm{Ch}}(G)$ is finite over $X = \operatorname{spf}(E^0)$ (Section 10). In Section 11, we recall the Hopkins-Kuhn-Ravenel generalised character theory, which relates $\mathbb{Q}\otimes E^0BG$ to the set $\Omega(G)$ of conjugacy classes of homomorphisms $\mathbb{Z}_p^n \to G$. We give a parallel (but less precise) relationship between $\mathbb{Q}\otimes C(E,G)$ and the set $\Omega_{\mathrm{Ch}}(G)$ of Λ-semiring homomorphisms $R^+(G) \to \mathbb{N}[(\mathbb{Q}_p/\mathbb{Z}_p)^n]$. These descriptions are related by a map $\kappa\colon \Omega(G) \to \Omega_{\mathrm{Ch}}(G)$. In Section 12, we compare $\Omega(G)$ and $\Omega_{\mathrm{Ch}}(G)$ with two other sets that are sometimes easier to understand. We next return to examples: in Section 14 we show that $X(\Sigma_4) = X_{\mathrm{Ch}}(\Sigma_4)$ at the prime 2, and in Section 15 we study $\Omega(G)$ and $\Omega_{\mathrm{Ch}}(G)$ when G is an extraspecial group at an odd prime. We then show that a certain approach which appears more precise actually captures no more information (Section 16). We conclude in Section 17 by proving a result in representation theory that was used in Section 9.

1. Notation and conventions

Fix a prime p. Throughout this paper, E will denote a p-local generalised cohomology theory with an associative and unital product. We write E^k for $E^k(\text{point})$, so E^* is a $\mathbb{Z}$-graded ring and E^0 is an ungraded ring. We assume that E has the following properties:

1. E^0 is a commutative complete local Noetherian ring, with maximal ideal $\mathfrak{m}$ say.
2. $E^k = 0$ whenever k is odd.
3. E^{-2} contains a unit (so $E^kX \simeq E^{k-2}X$ for all X).

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4. Either $p > 2$ and E is commutative, or $p = 2$ and E is quasi-commutative, which means that there is a natural derivation $Q\colon E^k X \to E^{k+1} X$ and an element $v \in E^{-2}$ such that $2v = 0$ and $ab - (-1)^{|a||b|} ba = vQ(a)Q(b)$ for all $a, b \in E^*X$.

There is one more condition, which needs some background explanation. Note that the quasi-commutativity condition means that whenever $E^1 X = 0$, the ring $E^0 X$ is commutative (in the usual ungraded sense.) In particular, $E^* = E^*(\text{point})$ is commutative. A collapsing Atiyah-Hirzebruch spectral sequence argument shows that

$$E^*\mathbb{C}P^\infty \simeq E^* \widehat{\otimes} H^*\mathbb{C}P^\infty = E^* \widehat{\otimes} \mathbb{Z}[\![y]\!] = E^*[\![y]\!],$$

where $y \in \widetilde{E}^2\mathbb{C}P^\infty$; it follows that E is complex-orientable. We can multiply y by a unit in E^{-2} to get an element $x \in \widetilde{E}^0\mathbb{C}P^\infty$ such that $E^*\mathbb{C}P^\infty = E^*[\![x]\!]$. We fix such an element x once and for all (although we will state our results in a form independent of this choice as far as possible). This gives rise in the usual way to a formal group law F over E^0.

5. In addition to the above properties, we will assume that the reduction of F modulo the maximal ideal of E^0 has height $n < \infty$.

In the language of [4], our conditions say that E is a $K(n)$-local admissible theory. The only difference is that previously we insisted that E should be commutative rather than quasi-commutative; the reader can easily check that everything in [4] goes through in the quasi-commutative case.

We will describe our results in the language of formal schemes. Most of the formal schemes that we consider have the form $\operatorname{spf}(R)$, where R is a complete local Noetherian E^0-algebra. For these the foundational setting discussed in [9] is satisfactory: one can regard the category of formal schemes as the opposite of the category of complete semilocal Noetherian rings and continuous homomorphisms. We also make some use of formal schemes such as $\operatorname{spf}(\prod_{k\in\mathbb{Z}} E^0[\![c_1, c_2, \dots]\!])$; a set of foundations covering these is developed in [8]. The older category of formal schemes embeds as a full subcategory of the newer one.

Definition 1.1. We let X be the formal scheme $\operatorname{spf}(E^0)$, and write $\mathbb{G}$ for the formal group $\operatorname{spf}(E^0\mathbb{C}P^\infty)$ over X. Note that our element $x \in \widetilde{E}^0\mathbb{C}P^\infty$ can be regarded as a coordinate on $\mathbb{G}$, with the property that

$$x(a+b) = F(x(a), x(b)) = x(a) +_F x(b).$$

Remark 1.2. Many of our constructions work with an arbitary formal group $\mathbb{G}$ over a formal scheme X; it is not usually necessary to assume that $\mathbb{G}$ comes from a cohomology theory, although that is the case of most interest for us.

We will let G denote a finite group. We write $e = e(G)$ for the exponent of G, in other words the least common multiple of the orders of the elements. We factor this in the form $e = p^v e' = p^{v(G)} e'(G)$, where $e' \neq 0 \pmod p$. We also choose a Sylow p-subgroup $P \leq G$.

2. Λ-(SEMI)RINGS

We will use the following definition:

Definition 2.1. A *semiring* is a set R equipped with the following structure.

- A commutative and associative addition law with neutral element (written as 0); we do not assume that there are additive inverses.
- A commutative and associative multiplication law with neutral element 1, which distributes over addition.

A Λ-*semiring* is a semiring R equipped with Maps $\lambda^k\colon R \to R$ for $k \geq 0$ satisfying $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ and $\lambda^k(x+y) = \sum_{k=i+j} \lambda^i(x)\lambda^j(y)$.

A Λ-ring is a Λ-semiring which has additive inverses.

The initial Λ-semiring is $\mathbb{N}$ and the initial Λ-ring is $\mathbb{Z}$; in both cases we have

$$\lambda^k(n) = \binom{n}{k} = n(n-1)\dots(n-k+1)/k!.$$

Definition 2.2. An $\mathbb{N}$-*augmented* Λ-semiring is a Λ-semiring R equipped with a homomorphism $\dim\colon R \to \mathbb{N}$ of Λ-semirings. A $\mathbb{Z}$-*augmented* Λ-ring is a Λ-ring R equipped with a homomorphism $\dim\colon R \to \mathbb{Z}$ of Λ-rings.

Example 2.3. Let $R^+(G)$ be the semiring of isomorphism classes of complex representations of G. It is well-known that this is a Λ-semiring with operations λ^k given by exterior powers. There is an augmentation $\dim\colon R^+(G) \to \mathbb{N}$ sending each representation to its dimension.

Example 2.4. Let A be an Abelian group, and let $\mathbb{N}[A]$ be the group semiring of A, in other words the set of expressions $\sum_a n_a[a]$ with $n_a \in \mathbb{N}$ and $n_a = 0$ for all but finitely many a. Equivalently, we have $\mathbb{N}[A] = \coprod_n A^n/\Sigma_n$. This has a canonical structure as a Λ-semiring, with

$$\lambda^k([a_1] + \ldots + [a_n]) = \sum_I [a_{i_1} + \ldots + a_{i_k}],$$

where the sum on the right runs over all lists $I = (i_1, \ldots, i_k)$ such that $1 \le i_1 < \ldots < i_k \le n$. There is an augmentation $\dim\colon \mathbb{N}[A] \to \mathbb{N}$ defined by

$$\dim([a_1] + \ldots + [a_n]) = n.$$

If A is finite and $A^* = \mathrm{Hom}(A, S^1)$ then $\mathbb{N}[A] = R^+(A^*)$ as Λ-semirings.

Example 2.5. For any space X, let $\mathrm{Vect}^+(X)$ denote the semiring of isomorphism classes of complex vector bundles over X. This is a Λ-semiring with operations as for $R^+(G)$. We will always allow vector bundles to have different dimensions over different components of the base, so we do not have a natural map $\dim\colon \mathrm{Vect}^+(X) \to \mathbb{Z}$. We write $\mathrm{Vect}^+_d(X)$ for the set of isomorphism classes of bundles all of whose fibres have dimension d, and we put $\mathrm{Pic}(X) = \mathrm{Vect}_1(X) \simeq H^2(X)$. This is an Abelian group, and there is an evident map $\mathbb{N}[\mathrm{Pic}(X)] \to \mathrm{Vect}^+(X)$. In the case $X = BG$, there is a well-known homomorphism $R^+(G) \to \mathrm{Vect}^+(BG)$ sending a representation V to the bundle $V \times_G EG$.

Remark 2.6. In the important examples of Λ-(semi)rings, some extra identities hold that relate the elements $\lambda^i\lambda^j(x)$ and $\lambda^i(xy)$ to the elements $\lambda^k(x)$ and $\lambda^l(y)$. For many purposes it would be preferable to take these identities as part of the definition of a Λ-(semi)ring. However, it turns out that this would make no difference for us and the identities are complicated (particularly in the semiring case) so we omit them. In Section 16 we will discuss an approach which is apparently even more precise, and show that it actually gives no more information than our approach using Λ-semirings without extra identities.

Remark 2.7. Let R^+ be a Λ-semiring, and let R be its Grothendieck completion, or in other words the group completion of R^+ considered as a monoid under addition. It is well-known that this can be made into a Λ-ring in a canonical way, and that any homomorphism from R^+ to a Λ-ring factors uniquely through R. Moreover, if R^+ is augmented over $\mathbb{N}$ then R is augmented over $\mathbb{Z}$.

Example 2.8. The Grothendieck completion of $R^+(G)$ is of course the ring $R(G)$ of virtual representations of G, and the completion of $\mathbb{N}[A]$ is the group ring $\mathbb{Z}[A]$. We write $\mathrm{Vect}(X)$ for the Grothendieck completion of $\mathrm{Vect}^+(X)$. It is well-known that the complex K-theory $K^0(X)$ is a $\mathbb{Z}$-augmented Λ-semiring and that there is a natural map $\mathrm{Vect}(X) \to K^0(X)$ which is an isomorphism whenever X is compact Hausdorff.

Remark 2.9. We will occasionally use the notation $\mathbb{Z}[A]^+ = \mathbb{N}[A]$ and $\mathbb{Z}[A]^+_d = A^d/\Sigma_d \subset \mathbb{N}[A]$.

3. Λ-(SEMI)RING SCHEMES

The theory of Λ-semirings is an instance of universal algebra: it is defined in terms of operations $\omega\colon R^k \to R$ with $k = 0, 1$ or 2, and identities between operations derived from these. It is thus formal to define the

notion of a Λ-semiring object in any category $\mathcal{C}$ with finite products: such a thing is an object $R \in \mathcal{C}$ equipped with maps

$$
\begin{aligned}
0, 1 \colon 1 &\to R \\
+, \times \colon R^2 &\to R \\
\lambda^k \colon R &\to R \qquad\qquad \text{for all } k \in \mathbb{N}
\end{aligned}
$$

making the evident diagrams commute. (Here the object $1 \in \mathcal{C}$ is the terminal object.) Similar remarks apply to Λ-rings.

Next, suppose that $\mathcal{C}$ has arbitrary coproducts such that the natural map

$$\coprod_{i,j} X_i \times Y_j \to \coprod_i X_i \times \coprod_j X_j$$

is always an isomorphism. We then have a product-preserving functor $S \mapsto \underline{S} := \coprod_{s \in S} 1$ from sets to $\mathcal{C}$, so $\underline{\mathbb{N}}$ is a Λ-semiring object in $\mathcal{C}$. Similarly, $\underline{\mathbb{Z}}$ is a Λ-ring object in $\mathcal{C}$.

Example 3.1. Take $\mathcal{C} = \overline{h}\mathcal{T}$, the homotopy category of unbased CW-complexes. We have a functor $\mathrm{Vect}^+(-)$ from $\overline{h}\mathcal{T}^{\mathrm{op}}$ to the category of Λ-semirings, which is represented by the space $\coprod_d BU(d)$. It follows by Yoneda's lemma that $\coprod_d BU(d)$ is a Λ-semiring in $\overline{h}\mathcal{T}$. Similarly, the functor $K^0(-)$ from $\overline{h}\mathcal{T}^{\mathrm{op}}$ to the category of Λ-rings is represented by the Λ-ring space $\mathbb{Z} \times BU$. Note that in this context the object $\underline{\mathbb{N}}$ is just the discrete space $\mathbb{N}$ and similarly for $\underline{\mathbb{Z}}$, so $\coprod_d BU(d)$ is augmented over $\underline{\mathbb{N}}$ and $\mathbb{Z} \times BU$ is augmented over $\underline{\mathbb{Z}}$.

Now let X be a formal scheme, and consider the category $\widehat{\mathcal{X}}_X$ of category of formal schemes over X in the sense of [8]. For simplicity we will assume that X is solid, which means that $X = \mathrm{spf}(\mathcal{O}_X)$ for some formal ring $\mathcal{O}_X$. Let $\mathcal{A}$ be the category of discrete $\mathcal{O}_X$-algebras, and let $\mathcal{F}$ be the category of functors from $\mathcal{A}$ to sets. The category $\widehat{\mathcal{X}}_X$ can be regarded as a subcategory of $\mathcal{F}$ (compare [8, Remark 2.1.5]), and the inclusion $\widehat{\mathcal{X}}_X \to \mathcal{F}$ preserves products.

We will refer to Λ-(semi)ring objects in $\widehat{\mathcal{X}}_X$ as Λ-(semi)ring schemes (suppressing the words "formal" and "over X" for brevity).

Let $\mathbb{G}$ be an ordinary formal group over X, in other words a commutative group object in $\widehat{\mathcal{X}}_X$ that is isomorphic in $\widehat{\mathcal{X}}_X$ to $\widehat{\mathbb{A}}^1_X = \mathrm{spf}(\mathcal{O}_X[\![x]\!])$. We can then define the schemes

$$
\begin{aligned}
\mathrm{Div}^+_d(\mathbb{G}) &= \mathbb{G}^d/\Sigma_d \\
\mathrm{Div}^+(\mathbb{G}) &= \coprod_{d \in \mathbb{N}} \mathrm{Div}^+_d(\mathbb{G}) \\
\mathrm{Div}_0(\mathbb{G}) &= \varinjlim_d \mathrm{Div}^+_d(\mathbb{G}) \\
\mathrm{Div}(\mathbb{G}) &= \underline{\mathbb{Z}} \times \mathrm{Div}_0(\mathbb{G}) \\
\mathrm{Div}_d(\mathbb{G}) &= \{d\} \times \mathrm{Div}_0(\mathbb{G}) \subset \mathrm{Div}(\mathbb{G}).
\end{aligned}
$$

More detailed definitions are given in [8, Section 5], where it is also explained how these formal schemes relate to the theory of divisors on $\mathbb{G}$. In [8, Proposition 6.2.7] it is observed that

1. $\mathrm{Div}^+(\mathbb{G})$ is the free commutative monoid object in $\mathcal{C}$ generated by $\mathbb{G}$.
2. $\mathrm{Div}(\mathbb{G})$ is the free commutative group object in $\mathcal{C}$ generated by $\mathbb{G}$.
3. $\mathrm{Div}_0(\mathbb{G})$ is the free commutative monoid object generated by $\mathbb{G}$ considered as a based object in $\mathcal{C}$, which is the same as the free commutative group object generated by $\mathbb{G}$ considered as a based object in $\mathcal{C}$.

Moreover, all these universal properties are stable under base change: if X' is a formal scheme over X then $\mathrm{Div}^+(\mathbb{G}) \times_X X'$ is the free commutative monoid in $\widehat{\mathcal{X}}_{X'}$ generated by $\mathbb{G} \times_X X'$ and so on.

Recall that $\mathcal{O}_{\mathbb{G}} = \mathcal{O}_X[\![x]\!]$ and thus $\mathcal{O}_{\mathbb{G}^d} = \mathcal{O}_X[\![x_1,\dots,x_d]\!]$. If c_k denotes the coefficient of t^{d-k} in $\prod_i(t-x_i)$ then $\mathcal{O}_{\mathrm{Div}_d^+(\mathbb{G})} = \mathcal{O}_X[\![c_1,\dots,c_d]\!]$ and $\mathcal{O}_{\mathrm{Div}^+(\mathbb{G})} = \prod_{d\geq 0}\mathcal{O}_X[\![c_1,\dots,c_d]\!]$. There are also isomorphisms

$$\mathcal{O}_{\mathrm{Div}_0(\mathbb{G})} = \mathcal{O}_X[\![c_1,c_2,\dots]\!]$$
$$\mathcal{O}_{\mathrm{Div}(\mathbb{G})} = \prod_{d\in\mathbb{Z}}\mathcal{O}_X[\![c_1,c_2,\dots]\!].$$

Using these, one sees that $\mathrm{Div}_d^+(\mathbb{G})$ is a closed subscheme of $\mathrm{Div}_d(\mathbb{G})$, and $\mathrm{Div}^+(\mathbb{G})$ is a closed subscheme of $\mathrm{Div}(\mathbb{G})$.

If E is an even periodic ring spectrum, $X = \mathrm{spec}(\pi_0 E)$ and $\mathbb{G} = \mathrm{spf}(E^0\mathbb{C}P^\infty)$ then there are natural isomorphisms

$$\mathrm{spf}(E^0BU(d)) = \mathrm{Div}_d^+(\mathbb{G})$$
$$\mathrm{spf}(E^0BU) = \mathrm{Div}_0(\mathbb{G})$$
$$\mathrm{spf}(E^0(\mathbb{Z}\times BU)) = \mathrm{Div}(\mathbb{G}).$$

This is just a translation of well-known calculations; details are given in [8, Section 8].

Proposition 3.2. *Let $\mathbb{G}$ be an ordinary formal group over a scheme X. Then $\mathrm{Div}^+(\mathbb{G})$ has a natural structure as a Λ-semiring scheme, and $\mathrm{Div}(\mathbb{G})$ has a natural structure as a Λ-ring scheme. Moreover, there is a canonical homomorphism $\dim\colon \mathrm{Div}(\mathbb{G}) \to \underline{\mathbb{Z}}$ of Λ-ring schemes, which sends $\mathrm{Div}_d(\mathbb{G})$ to d.*

Proof. Recall that $\mathcal{F}$ is the category of functors from discrete $\mathcal{O}_X$-algebras to sets. Define $R^+, R \in \mathcal{F}$ by $R^+(A) = \mathbb{N}[\mathbb{G}(A)]$ and $R(A) = \mathbb{Z}[\mathbb{G}(A)]$. It is clear that R^+ is a Λ-semiring object in $\mathcal{F}$, and R is a Λ-ring object.

There is an evident inclusion $j\colon \mathbb{G} = \mathrm{Div}_1^+(\mathbb{G}) \to \mathrm{Div}^+(\mathbb{G})$. As $\mathrm{Div}^+(\mathbb{G})$ is a commutative monoid scheme, the set $\mathrm{Div}^+(\mathbb{G})(A)$ is a commutative monoid for all $A \in \mathcal{A}$. As $R^+(A)$ is the free commutative monoid generated by the set $\mathbb{G}(A)$, there is a unique homomorphism $\phi^+\colon R^+(A) \to \mathrm{Div}^+(\mathbb{G})(A)$ extending j. These maps are natural in A so we get a map $\phi^+\colon R^+ \to \mathrm{Div}^+(\mathbb{G})$ in $\mathcal{F}$. If we interpret the colimits in $\widehat{\mathcal{X}}_X$ then we have $\mathrm{Div}^+(\mathbb{G}) = \coprod_d \mathbb{G}^d/\Sigma_d$; this translates to the statement that $\mathrm{Div}^+(\mathbb{G})$ is the initial example of a formal scheme in $\widehat{\mathcal{X}}_X$ equipped with a map $R^+ \to \mathrm{Div}^+(\mathbb{G})$ in $\mathcal{F}$. By similar arguments, we find that $\mathrm{Div}(\mathbb{G})$ is the initial example of a formal scheme over X with a map $\phi\colon R \to \mathrm{Div}(\mathbb{G})$ in $\mathcal{F}$. Moreover, one can check that the schemes $\mathrm{Div}^+(\mathbb{G})^k$ and $\mathrm{Div}(\mathbb{G})^k$ enjoy the evident analogous universal properties for all $k \geq 0$.

It now follows that there is a unique map $\times\colon \mathrm{Div}^+(\mathbb{G}) \times \mathrm{Div}^+(\mathbb{G}) \to \mathrm{Div}^+(\mathbb{G})$ making the following diagram commute:

$$\begin{tikzcd}
R^+ \times R^+ \arrow[r, "\times"] \arrow[d, "\phi^+\times\phi^+"'] & R^+ \arrow[d, "\phi^+"] \\
\mathrm{Div}^+(\mathbb{G}) \times \mathrm{Div}^+(\mathbb{G}) \arrow[r, "\times"'] & \mathrm{Div}^+(\mathbb{G}).
\end{tikzcd}$$

Similarly, all the other structure maps for the Λ-semiring structure on R^+ induce operations on $\mathrm{Div}^+(\mathbb{G})$, and one checks easily that this makes $\mathrm{Div}^+(\mathbb{G})$ into a Λ-semiring scheme. A similar argument works for $\mathrm{Div}(\mathbb{G})$. It is clear that there is a map $\dim\colon \mathrm{Div}(\mathbb{G}) \to \underline{\mathbb{Z}}$ as described. □

The above Λ-semiring structure can be made more explicit as follows. Let $c_{d,k} \in \mathcal{O}_X[\![x_1,\dots,x_d]\!]$ be defined by

$$\prod_{i=1}^{d}(t-x_i) = \sum_{i=0}^{d} c_{d,i}x^{d-i}.$$

Let $p_{d,e,k}(c_{d,1},\dots,c_{d,d},c'_{e,1},\dots,c'_{e,e})$ be defined by

$$\prod_{i=1}^{d}\prod_{j=1}^{e}(t-(x_i +_F x'_j)) = \sum_{k=0}^{de} p_{d,e,k}t^{de-k}.$$

Suppose $d, r \in \mathbb{N}$ and put $N = \binom{d}{r}$. Let $q_{d,r,k}(c_{N,1}, \dots, c_{N,N})$ be defined by

$$\prod_I (t - \sum_j^F x_{i_j}) = \sum_{k=0}^N q_{d,r,k} t^{N-k},$$

where the sum on the left runs over all lists $I = (i_1, \dots, i_r)$ such that $1 \leq i_1 < \dots < i_r \leq d$. Then the multiplication map

$$\times \colon \mathrm{Div}_d^+(\mathbb{G}) \times \mathrm{Div}_e^+(\mathbb{G}) \to \mathrm{Div}_{de}^+(\mathbb{G})$$

corresponds to the map

$$\mathcal{O}_X[\![c_{de,1}, \dots, c_{de,de}]\!] \to \mathcal{O}_X[\![c_{d,1}, \dots, c_{d,d}, c'_{1,e}, \dots, c'_{e,e}]\!]$$

(of formal $\mathcal{O}_X$-algebras) sending $c_{de,k}$ to $p_{d,e,k}$. Similarly, the map corresponding to $\lambda^r \colon \mathrm{Div}_d^+(\mathbb{G}) \to \mathrm{Div}_N^+(\mathbb{G})$ sends $c_{N,k}$ to $q_{d,r,k}$.

4. Adams operations

We now recall the theory of Adams operations in Λ-semirings; for a more detailed exposition see [6], for example.

Let R be a Λ-ring. For any $a \in R$ we can form the power series

$$\lambda_t(a) = \sum_{k \geq 0} \lambda^k(a)(-t)^k \in R[\![t]\!].$$

This is equal to 1 mod t and thus is invertible in $R[\![t]\!]$. It is easy to check that $\lambda_t(0) = 1$ and $\lambda_t(a+b) = \lambda_t(a)\lambda_t(b)$.

We next define

$$\psi_t(a) = -t\lambda_{-t}(a)^{-1}\, d\lambda_{-t}(a)/dt \in R[\![t]\!],$$

and let $\psi^k(a)$ be the coefficient of t^k in $\psi_t(a)$. This defines an additive map $\psi^k \colon R \to R$, called the k'th Adams operation.

Now consider the case $R = \mathbb{Z}[A]$ for some Abelian group A. It is not hard to see that

$$\psi^k(\sum_i n_i[a_i]) = \sum_i n_i[a_i]^k = \sum_i n_i[ka_i],$$

so ψ^k is just the map induced by the homomorphism $k.1_A \colon A \to A$. Thus, if A is actually a $\mathbb{Z}_{(p)}$-module or a $\mathbb{Z}_p$-module, then there is a natural way to define $\psi^k \colon \mathbb{Z}[A] \to \mathbb{Z}[A]$ for all $k \in \mathbb{Z}_{(p)}$ or $k \in \mathbb{Z}_p$ as appropriate. Moreover, we see that ψ^k is a ring homomorphism which preserves the semiring $\mathbb{Z}[A]^+$ and the subsets $\mathbb{Z}[A]^+_d$, and that $\psi^k\psi^j = \psi^{kj}$ and $\psi^k\lambda^j = \lambda^j\psi^k$.

Now consider the Λ-ring scheme $\mathrm{Div}(\mathbb{G})$. As our original definition of ψ^k is natural, we evidently get morphisms

$$\psi^k \colon \mathrm{Div}(\mathbb{G}) \to \mathrm{Div}(\mathbb{G})$$

of schemes. It is well-known that $\mathbb{G}$ is actually a $\mathbb{Z}_p$-module scheme, or in terms of our coordinate, that one can define the series $[k]_F(x)$ in a sensible way for all $k \in \mathbb{Z}_p$. This means that each ring $\mathbb{Z}[\mathbb{G}(A)]$ admits Adams operations ψ^k for all $k \in \mathbb{Z}_p$, with properties as above. The argument of Proposition 3.2 shows that

- We can define operations ψ^k on $\mathrm{Div}(\mathbb{G})$ for all $k \in \mathbb{Z}_p$, extending the definition given previously.
- These maps are maps of ring schemes, induced by the maps $k \colon \mathbb{G} \to \mathbb{G}$.
- We have $\psi^j\psi^k = \psi^{jk}$ for all $j, k \in \mathbb{Z}_p$, and $\psi^k\lambda^j = \lambda^j\psi^k$ for all $k \in \mathbb{Z}_p$ and $j \in \mathbb{N}$.
- The map ψ^k preserves $\mathrm{Div}_d(\mathbb{G})$, $\mathrm{Div}^+(\mathbb{G})$ and $\mathrm{Div}_d^+(\mathbb{G}) = \mathbb{G}^d/\Sigma_d$ for all d.

Lemma 4.1. *For any discrete $\mathcal{O}_X$-algebra A, the group $\mathbb{G}(A)$ is a p-torsion group.*

Proof. Our coordinate x gives an isomorphism $x \colon \mathbb{G}(A) \to \widehat{\mathbb{A}}^1(A) = \mathrm{Nil}(A)$ (the set of nilpotents in A). As p lies in the maximal ideal of $E^0 = \mathcal{O}_X$ and A is a discrete $\mathcal{O}_X$-algebra, we see that $p^r = 0 \in A$ for some r, and thus $[p^r](x)$ is divisible by x^2 in $A[\![x]\!]$. It follows that $[p^{rs}](x)$ is divisible by x^{2^s}. For any $a \in \mathbb{G}(A)$ we have $x(a)^{2^s} = 0$ for large s, so $x(p^{rs}a) = 0$ for large s, so $p^m a = 0$ for large m as required. $\square$

Lemma 4.2. *Let A be a discrete $\mathcal{O}_X$-algebra, and suppose we have a divisor $D \in \mathrm{Div}(\mathbb{G})(A)$. Then $\psi^k(D) = \dim(D)[0]$ whenever the p-adic valuation $v_p(k)$ is sufficiently large.*

Proof. First suppose that $D \in \mathrm{Div}_d^+(\mathbb{G})(A)$. We can then choose a faithfully flat map $A \to A'$ such that the image of D in $\mathrm{Div}_d^+(\mathbb{G})(A')$ has the form $\sum_{i=1}^d [a_i]$. The map $\mathrm{Div}_d^+(\mathbb{G})(A) \to \mathrm{Div}_d^+(\mathbb{G})(A')$ is automatically injective, so it suffices to show that for large m we have $\psi^{p^m}(\sum_i [a_i]) = \sum_i [p^m a_i] = d[0]$, which is immediate from the previous lemma.

Now suppose that $D \in \mathrm{Div}_d(\mathbb{G})(A)$. We can then write D in the form $D' - e[0]$ for some $D' \in \mathrm{Div}_{d+e}^+(\mathbb{G})(A)$ and we reduce easily to the previous case.

Finally, consider a general divisor $D \in \mathrm{Div}_d(\mathbb{G})(A)$, which need not have constant dimension. Instead, we have a splitting $A = A_1 \times \ldots \times A_r$ giving a bijection $\mathrm{Div}(\mathbb{G})(A) = \prod_i \mathrm{Div}(\mathbb{G})(A_i)$ under which D becomes an r-tuple $(D_1, \ldots, D_r)$ with $D_i \in \mathrm{Div}_{d_i}^+(\mathbb{G})(A_i)$ for some integers d_i. This means that $\dim(D)$ becomes $(d_1, \ldots, d_r)$ under the bijection $\underline{\mathbb{Z}}(A) = \prod_i \underline{\mathbb{Z}}(A_i)$. The cases considered previously imply that

$$\psi^k D = (\psi^k D_1, \ldots, \psi^k D_r) = (d_1[0], \ldots, d_r[0]) = \dim(D)[0]$$

when $v_p(k) \gg 0$, as required. □

Now consider instead the Λ-ring $R(G)$. In this case we can define Adams operations ψ^k for $k \in \mathbb{N}$, and it is well-known that in terms of characters we have

$$\chi_{\psi^k V}(g) = \chi_V(g^k).$$

As a virtual representation is determined by its character and $\dim(V) = \chi_V(1)$, it follows easily that ψ^k is a degree-preserving map of Λ-rings and that $\psi^j\psi^k = \psi^{jk}$. Moreover, if e is the exponent of G (in other words, least common multiple of the orders of the elements) then ψ^k depends only on the congruence class of k modulo e. If k is coprime to e (or equivalently, to $|G|$), it follows that $\psi^k\psi^j = 1$ for some j, so ψ^k is an isomorphism. In this case the map $g \mapsto g^k$ is a bijection, and it follows easily that ψ^k preserves the usual inner product on $R(G)$. A virtual representation V is an irreducible honest representation iff $\chi_V(1) > 0$ and $\langle V, V\rangle = 1$, and it follows that ψ^k sends irreducibles to irreducibles and thus sends $R_d^+(G)$ to $R_d^+(G)$. (Compare [7, Exercise 9.4].)

However, if k is not coprime to e then ψ^k need not preserve $R^+(G)$. For example, take $G = \Sigma_3$, let ϵ be the nontrivial one-dimensional representation, and let ρ be the irreducible two-dimensional representation. We then have $\psi^2(\rho) = \rho + 1 - \epsilon \notin R^+(G)$.

Lemma 4.3. *Let p^v be the p-part of the exponent of G. Then for any homomorphism $f\colon R(G) \to \mathrm{Div}(\mathbb{G})(A)$ of Λ-rings and any $V \in R_d(G)$ we have $f(V) \in \mathrm{Div}_d(\mathbb{G})$ and $\psi^{p^v} f(V) = d[0]$.*

Proof. Let the exponent of G be $e = p^v e'$, where e' is coprime to p. The map $\psi^{e'}\colon \mathrm{Div}(\mathbb{G}) \to \mathrm{Div}(\mathbb{G})$ is an isomorphism and fixes $d[0]$, and $\psi^{e'}\psi^{p^v} = \psi^e$ so it suffices to show that $\psi^e f(V) = d[0]$. To see this note that $\psi^e f(V) = f(\psi^e V)$ and $\chi_{\psi^e V}(g) = \chi_V(g^e) = \chi_V(1) = d$ for all g, so $\psi^e V$ is the trivial representation of rank d. As f is a ring map, we have $f(\psi^e V) = f(d) = d[0]$, as required.

This implies that $\psi^{p^k} f(V) = d[0]$ for $k \gg 0$ but Lemma 4.2 says that $\psi^{p^k} f(V) = \dim(f(V))[0]$ for $k \gg 0$, so $\dim(f(V)) = d$, so $f(V) \in \mathrm{Div}_d(\mathbb{G})$ as claimed. □

5. Chern approximations

Definition 5.1. Let G be a finite group, and let A be a discrete $\mathcal{O}_X$-algebra. We define a functor $X_{\mathrm{Ch}}(G)$ from discrete $\mathcal{O}_X$-algebras to sets by

$$X_{\mathrm{Ch}}(G)(A) = \{ \text{ homomorphisms } R^+(G) \to \mathrm{Div}^+(\mathbb{G})(A) \text{ of } \Lambda\text{-seimirings } \}.$$

We write $C(E, G)$ for the ring $\mathcal{O}_{X_{\mathrm{Ch}}(G)}$ of natural transformations from $X_{\mathrm{Ch}}(G)$ to the forgetful functor $\mathbb{A}^1$. We also put $X(G) = \mathrm{spf}(E^0 BG)$. We refer to $C(E, G)$ as the Chern approximation to $E^0 BG$, and to $X_{\mathrm{Ch}}(G)$ as the Chern approximation to $X(G)$.

Remark 5.2. We say that a homomorphism $f\colon R(G) \to \mathrm{Div}(\mathbb{G})(A)$ of Λ-rings is *positive* if $f(R^+(G)) \subseteq \mathrm{Div}^+(\mathbb{G})(A)$. It is clear from Remark 2.7 that $X_{\mathrm{Ch}}(G)(A)$ bijects naturally with the set of positive homomorphisms $R(G) \to \mathrm{Div}(\mathbb{G})(A)$, and we will implicitly use this identification where convenient. We also see from Lemma 4.3 that positive homomorphisms satisfy $f(R_d^+(G)) \subseteq \mathrm{Div}_d^+(\mathbb{G})(A)$.

Proposition 5.3. *The functor $X_{Ch}(G)$ is a formal scheme over X. The ring $C(E,G) = \mathcal{O}_{X_{Ch}(G)}$ is a quotient of a formal power series ring in finitely many variables over $\mathcal{O}_X$ (and thus is a complete Noetherian local ring).*

Proof. Let $V_1,\dots,V_h$ be the irreducible representations of G, and let $d_1,\dots,d_h$ be their degrees. We assume that these are ordered so that V_1 is the trivial representation of rank one. There are then natural numbers m_{ijk} and l^r_{ij} for $r\geq 0$ and $1\leq i,j,k\leq h$ such that

$$V_i\otimes V_j \simeq \bigoplus_k m_{ijk}.V_k$$

$$\lambda^r V_i \simeq \bigoplus_j l^r_{ij}.V_j$$

(Here $m.W$ means the direct sum of m copies of W.)

To give a homomorphism $f\colon R^+(G)\longrightarrow \mathrm{Div}^+(\mathbb{G})(A)$ is the same as to give divisors $D_i = f(V_i)\in \mathrm{Div}^+_{d_i}(\mathbb{G})$ for $i=1,\dots,h$ such that

$$D_iD_j = \sum_{ijk} m_{ijk}D_k$$

$$\lambda^r D_i = \sum_j l^r_{ij}D_j$$

This exhibits $X_{\mathrm{Ch}}(G)(A)$ as the equaliser of a pair of maps from $\prod_{i=1}^h \mathrm{Div}_{d_i}(\mathbb{G})$ to

$$\prod_{i,j}\mathrm{Div}_{d_id_j}(\mathbb{G})\times\prod_{r,i}\mathrm{Div}_{\binom{d_i}{r}}(\mathbb{G}).$$

In particular, this is a pair of maps between formal schemes over X, so the equaliser is a formal scheme over X.

More explicitly, we have $X_{\mathrm{Ch}}(G) = \mathrm{spf}(C(E,G))$, where $C(E,G)$ is defined as follows. We start with $\mathcal{O}_X$ and adjoin power series variables c_{ik} for $i=1,\dots,h$ and $k=1,\dots,d_i$, and put $c_{i0}=1$. We then put $f_i(t) = \sum_{k=0}^{d_i} c_{ik}t^{d_i-k}$ and impose the relations obtained by equating coefficients in the following identities between polynomials:

$$\sum_{a=0}^{d_id_j} p_{d_i,d_j,a}(c_{i*},c_{j*})t^{d_id_j-a} = \prod_k f_k(t)^{m_{ijk}}$$

$$\sum_{a=0}^{\binom{d_i}{r}} q_{d_i,r,a}(c_{i*})t^{\binom{d_i}{r}-a} = \prod_j f_j(t)^{l^r_{ij}}$$

The resulting quotient ring is $C(E,G)$. $\square$

We next explain how to compare $X_{\mathrm{Ch}}(G)$ to $X(G)$. Let $\overline{\mathcal{G}}$ be the category whose objects are Lie groups, and whose morphisms are the conjugacy classes of continuous homomorphisms. We then have a natural map

$$R^+(G) = \coprod_d \overline{\mathcal{G}}(G,U(d)) \xrightarrow{B} \overline{h}\mathcal{T}(BG,\coprod_d BU(d)) \xrightarrow{\mathrm{spf}(E^0(-))} \widehat{\mathcal{X}}_X(X(G),\mathrm{Div}^+(\mathbb{G})).$$

By taking adjoints, we obtain a map $X(G)\longrightarrow \mathrm{Map}(R^+(G),\mathrm{Div}^+(\mathbb{G}))$, and one checks easily that this actually lands in the subscheme $X_{\mathrm{Ch}}(G)\subset \mathrm{Map}(R^+(G),\mathrm{Div}^+(\mathbb{G}))$ of Λ-semiring homomorphisms. We thus have a natural map

$$\theta_G\colon X(G)\longrightarrow X_{\mathrm{Ch}}(G).$$

In terms of our explicit description of $C(E,G)$, the map $\theta^*\colon C(E,G)\longrightarrow E^0BG$ sends c_{ik} to the k'th Chern class of the representation V_i.

It is natural to ask whether a homomorphism $f\colon R(G)\longrightarrow \mathrm{Div}(\mathbb{G})(A)$ of Λ-rings is automatically positive. We next show that we always have $f(R^+_1(G))\subseteq \mathrm{Div}^+_1(\mathbb{G})\simeq\mathbb{G}$, but the corresponding claim for $d>1$ seems to be false.

Proposition 5.4. *If $D \in \mathrm{Div}_1(\mathbb{G})(A)$ and $\lambda^k(D) = 0$ for all $k > 1$ then $D \in \mathrm{Div}_1^+(\mathbb{G})(A)$.*

Proof. We can write $D = E - e[0]$ for some $e \geq 0$ and $E \in \mathrm{Div}_{e+1}^+(\mathbb{G})(A)$. Put $D' = \lambda^{e+1}E \in \mathrm{Div}_1^+(\mathbb{G})$. We have $E = D + e[0]$ so

$$D' = \sum_{i+j=e+1} \lambda^i(D)\lambda^j(e[0]) = \lambda^1(D)\lambda^e(e[0]) = D,$$

so $D \in \mathrm{Div}_1^+(\mathbb{G})$ as claimed. □

Corollary 5.5. *If $L \in R_1^+(G)$ and $f\colon R(G) \longrightarrow \mathrm{Div}(\mathbb{G})(A)$ is a map of Λ-rings then $f(L) \in \mathrm{Div}_1^+(\mathbb{G})$.*

Proof. Clearly $\lambda^k L = 0$ for $k > 1$ so $\lambda^k f(L) = 0$ for $k > 1$, and $f(L) \in \mathrm{Div}_1(\mathbb{G})(A)$ by Lemma 4.3 so $f(L) \in \mathrm{Div}_1^+(\mathbb{G})$ by the proposition. □

Proposition 5.6. *For suitable formal groups $\mathbb{G}$ and rings A, there exist divisors $D \in \mathrm{Div}_2(\mathbb{G})(A)$ such that $\lambda^k D = 0$ for $k > 2$ but $D \notin \mathrm{Div}_2^+(\mathbb{G})$.*

Proof. We will assume that $\mathcal{O}_X = \mathbb{F}_2$, so $p = 2$. Suppose that $a, b \in \mathbb{G}(A)$ and $2a = 2b = 0$. Put $c = a + b$ so $2a = 2b = 2c = a+b+c = 0$, and put $E = [a]+[b]+[c]$ and $D = E-[0]$. Then $\lambda^2 E = [a+b]+[b+c]+[c+a] = [c]+[a]+[b] = E$ and $\lambda^3 E = [0] = 1$ so $\lambda_t(E) = 1+tE+t^2E+t^3 = (1+tD+t^2)(1+t)$, so $\lambda_t(D) = 1+tD+t^2$. Thus $\lambda^k D = 0$ for $k > 2$. If D is in $\mathrm{Div}_2^+(\mathbb{G})(A)$ we must have $x(a)x(b)x(c) = c_3(E) = c_3(D+[0]) = 0$. Note also that $x(c) = x(a-b) = x(a) -_F x(b)$, which is a unit multiple of $x(a)-x(b)$, so the condition is equivalent to $x(a)^2x(b) = x(a)x(b)^2$. The universal example for A is $\mathcal{O}_X[\![y,z]\!]/([2](y),[2](z)) = \mathbb{F}_2[y,z]/(y^{2^n}, z^{2^n})$ (where $y = x(a), z = x(b)$). Clearly in this case we have $y^2z \neq yz^2$ so $D \notin \mathrm{Div}_2^+(\mathbb{G})$. □

6. The group Σ_3

In this section we work through the case where $G = \Sigma_3$ and E is the 2-periodic version of Morava K-theory at the prime 3 with height 2. Many constructions discussed here will be generalised later. Recall that the coefficient ring is $E^* = \mathbb{F}_3[u, u^{-1}]$, where $|u| = -2$.

We have a coordinate x on $\mathbb{G}$ such that

$$\begin{aligned} x(-a) &= [-1](x(a)) = -x(a) \\ x(3a) &= [3](x(a)) = x(a)^9 \\ x(a+b) &= x(a) +_F x(b) = x(a) + x(b) \pmod{x(a)^3x(b)^3} \end{aligned}$$

for all $a, b \in \mathbb{G}$. (The first equation is true because the formal group law F associated to E has an integral lift whose logarithm $\log_F(x) = \sum_k x^{9^k}/3^k$ satisfies $\log_F(-x) = -\log_F(x)$. The second is well-known, and the third follows from [9, Lemma 80].)

Define $y, z\colon \mathrm{Div}_2^+(\mathbb{G}) \longrightarrow \mathbb{A}^1$ by $y([a]+[b]) = x(a)x(b)$ and

$$z([a]+[b]) = x(\lambda^2([a]+[b])) = x(a+b) = x(a) +_F x(b)$$

(which is a unit multiple of $x(a)+x(b)$). One checks that that $\mathcal{O}_{\mathrm{Div}_2^+(\mathbb{G})} = \mathbb{F}_3[\![y,z]\!]$. If we let $Z = \mathrm{SDiv}_2^+(\mathbb{G})$ be the scheme of divisors $D \in \mathrm{Div}_2^+(\mathbb{G})$ such that $\lambda^2(D) = [0]$ then it follows that $\mathcal{O}_Z = \mathbb{F}_3[\![y,z]\!]/z = \mathbb{F}_3[\![y]\!]$. There is an evident map $\delta\colon \mathbb{G} \longrightarrow Z$ defined by $\delta(b) = [b] + [-b]$, and $y(\delta(b)) = x(b)x(-b) = -x(b)^2$ so the map $\delta^*\colon \mathbb{F}_3[\![y]\!] \longrightarrow \mathbb{F}_3[\![x]\!]$ sends y to $-x^2$. In particular, we see that δ is finite and faithfully flat, with degree two.

Next, note that

$$\delta(b)^2 = [2b] + [-2b] + 2[0] = \psi^2(\delta(b)) + 2[0];$$

as δ is faithfully flat, it follows that $D^2 = \psi^2 D + 2[0]$ for any $D \in Z$.

Let Y be the scheme of divisors $D \in Z$ such that $\psi^2(D) = D$. To analyse this, note that

$$x(2b) = x(-b+3b) = [-1](x(b)) +_F [3](x(b)) = -x(b) + x(b)^9 \pmod{x(b)^{12}},$$

so

$$-x(2b)^2 = -x(b)^2 - x(b)^{10} \pmod{x(b)^{12}},$$

or in other words
$$y(\psi^2\delta(b)) = y(\delta(b)) - y(\delta(b))^5 \pmod{y(\delta(b))^6}.$$
As δ is faithfully flat, we deduce that
$$y(\psi^2(D)) = y(D) - y(D)^5 \pmod{y(D)^6}$$
for all $D \in Z$. It follows that $(\psi^2)^*y - y$ is a unit multiple of y^5 in $\mathbb{F}_3[\![y]\!]$ and thus that $\mathcal{O}_Y = \mathbb{F}_3[y]/y^5$.

The character table of $G = \Sigma_3$ is

	1	ϵ	σ
1^3	1	1	2
1.2	1	-1	0
3	1	1	-1

From this we see that
$$R(G) = \mathbb{Z}[\epsilon, \sigma]/(\epsilon^2 - 1, \epsilon\sigma - \sigma, \sigma^2 - \sigma - 1 - \epsilon).$$
The only interesting λ-operation is that $\lambda^2(\sigma) = \epsilon$.

Let $f\colon R^+(G) \to \mathrm{Div}^+(\mathbb{G})(A)$ be a Λ-semiring homomorphism, in other words a point of $X_{\mathrm{Ch}}(G)$. As $\mathrm{Div}_1^+(\mathbb{G}) \simeq \mathbb{G}$, there is a unique point $a \in \mathbb{G}$ such that $f(\epsilon) = [a]$. We also write $D = f(\sigma) \in \mathrm{Div}_2^+(\mathbb{G})$. As f is a map of Λ-semirings, these satisfy
$$\begin{aligned} [2a] &= [a]^2 = f(\epsilon^2) = f(1) = [0] \\ [a]D &= f(\epsilon\sigma) = f(\sigma) = D \\ D^2 &= f(\sigma^2) = f(\sigma + 1 + \epsilon) = D + [0] + [a] \\ \lambda^2 D &= f(\lambda^2\sigma) = f(\epsilon) = [a]. \end{aligned}$$
As we work mod 3, the map $2\colon \mathbb{G} \to \mathbb{G}$ is an isomorphism so the first equation gives $a = 0$, so the second equation is automatic and the last equation says that $D \in Z$. Thus, the third equation becomes
$$\psi^2 D + 2[0] = D^2 = D + 2[0].$$
The semiring $\mathrm{Div}^+(\mathbb{G})$ embeds in the ring $\mathrm{Div}(\mathbb{G})$ so we can cancel to see that $\psi^2(D) = D$, so $D \in Y$. We can thus define a map $\chi\colon X_{\mathrm{Ch}}(G) \to Y$ by $\chi(f) = f(\sigma)$. One can check that the whole argument is reversible, so χ is an isomorphism and
$$C(E, G) = \mathcal{O}_{X_{\mathrm{Ch}}(G)} \simeq \mathcal{O}_Y = \mathbb{F}_3[y]/y^5.$$

We also have a short exact sequence $C_3 \to G \to C_2$ leading to an Atiyah-Hirzebruch-Serre spectral sequence
$$H^p(C_2; E^q BC_3) \Longrightarrow E^{p+q} BG.$$
We have $E^* BC_3 = \mathbb{F}_3[u^{\pm 1}][x]/x^9$ (where $|u| = 2$ and $|x| = 0$), and C_2 acts on this by $u \mapsto u$ and $x \mapsto [-1](x) = -x$. Because C_2 has order coprime to 3 we see that the spectral sequence collapses to an isomorphism
$$E^* BG = (E^* BC_3)^{C_2} = E^*[y]/y^5,$$
where $y = -x^2$. After some comparison of definitions we see that the map $\theta_G\colon X(G) \to X_{\mathrm{Ch}}(G)$ is an isomorphism.

7. The Abelian case

Theorem 7.1. *If G is Abelian then $\theta_G\colon X(G) \to X_{Ch}(G)$ is an isomorphism.*

Proof. Put $G^* = \mathrm{Hom}(G, S^1)$, so $R^+(G) = \mathbb{N}[G^*]$, and let A be an $\mathcal{O}_X$-algebra. If $f\colon \mathbb{N}[G^*] \to \mathrm{Div}^+(\mathbb{G})(A)$ is a Λ-semiring homomorphism, then f induces a group homomorphism $f'\colon G^* = R_1^+(G) \to \mathbb{G}(A) = \mathrm{Div}_1^+(\mathbb{G})(A)$. Conversely, given a group homomorphism $f'\colon G^* \to \mathbb{G}(A)$ we get a map $R^+(G) = \mathbb{N}[G^*] \to \mathbb{N}[\mathbb{G}(A)]$ of Λ-semirings. We can compose this with the map $\mathbb{N}[\mathbb{G}(A)] \to \mathrm{Div}^+(\mathbb{G})(A)$ in the proof of Proposition 3.2 to get a map $R^+(G) \to \mathrm{Div}^+(\mathbb{G})(A)$, or in other words a point of $X_{\mathrm{Ch}}(G)(A)$. One checks that these constructions give a bijection $\mathrm{Hom}(G^*, \mathbb{G}(A)) \simeq X_{\mathrm{Ch}}(G)(A)$, or equivalently an isomorphism

$\mathrm{Hom}(G^*, \mathbb{G}) \simeq X_{\mathrm{Ch}}(G)$. There is also an isomorphism $X(G) \simeq \mathrm{Hom}(G^*, \mathbb{G})$ (see [4, Proposition 2.9]), so we have an isomorphism $X(G) \simeq X_{\mathrm{Ch}}(G)$. A straightforward comparison of definitions shows that this is the same as θ_G. □

8. The height one case

In this section we choose a prime p and let E be the p-complete complex K-theory spectrum. We thus have $X = \mathrm{spf}(\mathbb{Z}_p)$, so discrete $\mathcal{O}_X$-algebras are just p-torsion rings. We also have $\mathbb{G} = \widehat{\mathbb{G}}_m$, so

$$\mathbb{G}(A) = \{u \in A^\times \mid 1 - u \text{ is nilpotent }\}.$$

Theorem 8.1. *If E is the p-adic complex K-theory spectrum and G is a p-group then the map $\theta_G \colon X(G) \to X_{Ch}(G)$ is an isomorphism.*

The rest of this section constitutes the proof. We fix a p-group G and write $\theta = \theta_G$ for brevity.

The first ingredient is the Atiyah-Segal completion theorem. In the case of a p-group, this says that

$$\mathcal{O}_{X(G)} = E^0 BG = \mathbb{Z}_p \otimes R(G).$$

We know that $\mathrm{Div}(\widehat{\mathbb{G}}_m)$ is the free ring scheme generated by the group scheme $\widehat{\mathbb{G}}_m$. Recall that the affine line $\mathbb{A}^1$ is just the forgetful functor from p-torsion rings to sets. This is a ring scheme in a natural way, and it contains $\widehat{\mathbb{G}}_m$ as a subgroup of its group of units. We thus have a ring map

$$\xi \colon \mathrm{Div}(\widehat{\mathbb{G}}_m) \to \mathbb{A}^1$$

extending the inclusion of $\widehat{\mathbb{G}}_m$ in $\mathbb{A}^1$. If $D = \sum_i n_i[u_i] \in \mathrm{Div}(\widehat{\mathbb{G}}_m)(A)$ then $\xi(D) = \sum_i n_i u_i \in \mathbb{A}^1(A) = A$.

Next recall that $\mathrm{Div}_d^+(\widehat{\mathbb{G}}_m) = \widehat{\mathbb{G}}_m^d/\Sigma_d$, so to describe a function $f \colon \mathrm{Div}_d^+(\widehat{\mathbb{G}}_m) \to \mathbb{A}^1$ it suffices to give the symmetric function $\overline{f} \colon \widehat{\mathbb{G}}_m^d \to \mathbb{A}^1$ such that

$$f(\sum_{i=1}^d [u_i]) = \overline{f}(u_1, \dots, u_d).$$

Let $\sigma_j \colon \mathbb{A}^d \to \mathbb{A}^1$ be the j'th elementary symmetric function and define

$$\begin{aligned} c_j(\sum_i [u_i]) &= \sigma_j(1 - u_1, \dots, 1 - u_d) \\ a_j(\sum_i [u_i]) &= \sigma_j(u_1, \dots, u_d) \\ a'_j(\sum_i [u_i]) &= a_j(\sum_i [u_i]) - \tbinom{d}{j}. \end{aligned}$$

Recall that $\mathcal{O}_{\mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)}$ is the set of all maps $\mathrm{Div}_d^+(\widehat{\mathbb{G}}_m) \to \mathbb{A}^1$, so c_j, a_j and a'_j can be viewed as elements of this ring. The function $u \mapsto 1 - u$ is a coordinate on the formal group $\widehat{\mathbb{G}}_m$ so a well-known argument gives an isomorphism

$$\mathcal{O}_{\mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)} = \mathbb{Z}_p[\![c_1, \dots, c_d]\!].$$

It is not hard to deduce that

$$\mathcal{O}_{\mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)} = \mathbb{Z}_p[\![a'_1, \dots, a'_d]\!],$$

which is the completion of the ring $\mathbb{Z}_p[a_1, \dots, a_d]$ at the ideal generated by the elements $a'_i = a_i - \binom{d}{i}$. Note also that $a_1(D) = \xi(D)$ for $D \in \mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)$.

Lemma 8.2. *For any divisor $D \in \mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)$ we have $a_j(D) = \xi(\lambda^j(D))$.*

Proof. We may assume that $D = \sum_i [u_i]$ for some elements $u_1, \dots, u_d \in \widehat{\mathbb{G}}_m$. For any $I \subseteq \{1, \dots, d\}$ with $|I| = j$ we put $u_I = \prod_{i \in I} u_i$. We then have $\lambda^j D = \sum_I [u_I]$ and thus

$$\xi\lambda^j(D) = \sum_I u_I = \sigma_j(u_1, \dots, u_d) = a_j(D).$$

□

Let V be a complex vector bundle over a space Z with associated projective bundle PV, and let $\mathbb{D}(V) = \mathrm{spf}(E^0PV)$ be the corresponding divisor on $\mathbb{G}$ over $\mathrm{spf}(E^0Z)$. If $V = \bigoplus_{i=1}^d L_i$ for some complex line bundles L_i then each L_i can be regarded as an element of $E^0Z = K^0(Z;\mathbb{Z}_p)$, and one sees easily that $\mathbb{D}(V) = \sum_i [L_i]$ so $a_j(\mathbb{D}(V)) = \sigma_j(L_1,\dots,L_d) = \lambda^j(V)$. By the splitting principle we see that

$$\xi\lambda^j(\mathbb{D}(V)) = a_j(\mathbb{D}(V)) = \lambda^j(V)$$

even when V does not split as a sum of line bundles.

Now consider the case $Z = BG$ and suppose that V comes from a representation of G, which we also call V. Suppose we have a point $x \in X(G)(A)$ for some p-torsion ring A, corresponding to a ring map $\hat{x}\colon E^0BG \to A$. One can now check from the definitions that $\theta(x)(V) = \hat{x}_*(\mathbb{D}(V))$, so $f(\theta(x)(V)) = \hat{x}(f(\mathbb{D}(V)))$ for any $f \in \mathcal{O}_{\mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)}$. In particular, we have

$$a_j(\theta(x)(V)) = \hat{x}(a_j(\mathbb{D}(V))) = \hat{x}(\lambda^j(V)).$$

We can now construct the map $\zeta\colon X_{\mathrm{Ch}}(G) \to X(G)$ that will turn out to be inverse to θ. Let A be a p-torsion ring. A positive homomorphism $f \in X_{\mathrm{Ch}}(G)(A)$ gives rise to a homomorphism $\xi_A \circ f\colon R(G) \to \mathbb{A}^1(A) = A$, which factors canonically through $\mathbb{Z}_p \otimes R(G) = E^0BG = \mathcal{O}_{X(G)}$ because A is a p-torsion ring. This gives a continuous homomorphism $\mathcal{O}_{X(G)} \to A$, or in other words a point of $X(G)(A)$, which we call $\zeta(f)$. This construction gives a natural map $\zeta\colon X_{\mathrm{Ch}}(G) \to X(G)$, as required.

Suppose we start with a point $x \in X(G)(A)$, corresponding to a ring map $\hat{x}\colon \mathbb{Z}_p \otimes R(G) = E^0BG \to A$. We need to check that $\zeta\theta(x) = x$, or equivalently that $\xi_A(\theta(x)(V)) = \hat{x}(V)$ for all $V \in \mathbb{Z}_p \otimes R(G)$, and it will suffice to do this for all honest representations $V \in R_d^+(G)$ for all d. In that context we have $\xi = a_1$ so

$$\xi_A(\theta(x)(V)) = a_1(\theta(x)(V)) = \hat{x}(\lambda^1(V)) = \hat{x}(V)$$

as required. Thus $\zeta\theta = 1_{X(G)}$.

Suppose instead that we start with a point $f \in X_{\mathrm{Ch}}(G)(A)$, in other words a positive homomorphism $f\colon R^+(G) \to \mathrm{Div}^+(\widehat{\mathbb{G}}_m)(A)$. We need to check that $(\theta(\zeta(f)))(V) = f(V) \in \mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)(A)$ for all $V \in R_d^+(G)$. To see this, note that

$$\begin{aligned} a_j(\theta(\zeta(f))(V)) &= \widehat{\zeta(f)}(\lambda^j(V)) \\ &= \xi(f(\lambda^j(V))) \\ &= \xi(\lambda^j(f(V))) \\ &= a_j(f(V)). \end{aligned}$$

It follows that $a'_j(\theta(\zeta(f))(V)) = a'_j(f(V))$ and the functions a'_j generate $\mathcal{O}_{\mathrm{Div}_d^+(\widehat{\mathbb{G}}_m)}$ so $\theta(\zeta(f))(V) = f(V)$, as required. This shows that $\theta\zeta = 1_{X_{\mathrm{Ch}}(G)}$, so θ is an isomorphism as claimed.

9. Reduction to the Sylow subgroup

By a well-known transfer argument, if P is a Sylow p-subgroup of G then the restriction map $E^0BG \to E^0BP$ is injective, and similar methods give some control over the image. In this section we develop some analogous results for the approximation $C(E,G)$ to E^0BG. Let I be the kernel of the restriction map $R(G) \to R(P)$ (which is independent of the choice of P). Note that $R(G)/I$ is isomorphic to the image of the restriction map, so it is a free Abelian group of finite rank and it inherits a Λ-ring structure.

Proposition 9.1. *Any Λ-ring homomorphism $f\colon R(G) \to \mathrm{Div}(\mathbb{G})(A)$ factors through $R(G)/I$.*

Proof. As usual, we let the exponent of G be $e = p^v e'$, where e' is coprime to p. If $V \in I$ and $g \in G$ has p-power order then g is conjugate to an element of P and thus $\chi_V(g) = 0$. If g is an arbitrary element of g then the order of g divides $p^v e'$ so the order of $g^{e'}$ divides p^v, so $\chi_V(g^{e'}) = 0$. This proves that $\psi^{e'}(V) = 0$, so $\psi^{e'}(f(V)) = 0$ in $\mathrm{Div}(\mathbb{G})(A)$. However, the action of $\psi^{e'}$ on $\mathrm{Div}(\mathbb{G})(A)$ is induced by the action of e' on $\mathbb{G}$, which is invertible because e' is coprime to p. This implies that $f(V) = 0$ as claimed. $\square$

Corollary 9.2. *If $|G|$ is prime to p then $X_{Ch}(G) = X(G) = X$.* $\square$

Corollary 9.3. *$X_{Ch}(G)$ is a closed subscheme of the scheme of Λ-ring maps $R(G)/I \to \mathrm{Div}(G)$.* $\square$

Remark 9.4. The disadvantage of this point of view is that the positivity conditions $f(R^+(G)) \subseteq \mathrm{Div}^+(\mathbb{G})$ become less visible when we work with $R(G)/I$. However, the situation simplifies again if we assume that P is normal in G. In that case, it is known that the map $R^+(G) \to R^+(P)^G$ is surjective; this was first proved by Gallagher [2], and we will give an alternative proof in Section 17. The sums of G-orbits of irreducibles in $R^+(P)$ give a canonical system of generators for $R^+(P)^G$, which we can lift to get representations $\rho_1, \dots, \rho_t$ of G say. Let d_i be the degree of ρ_i. If $\rho \in R^+_d(G)$ then $\mathrm{res}^G_P(\rho) \in R^+_d(P)^G$ so $\mathrm{res}^G_P(\rho) \simeq \sum_i m_i \,\mathrm{res}^G_P(\rho_i)$ for some integers $m_i \geq 0$ with $\sum_i m_i d_i = d$, so $\rho - \sum_i m_i \rho_i \in I$. Thus, for any map $f\colon R(G) \to \mathrm{Div}(\mathbb{G})$ of Λ-rings we have $f(\rho) = \sum_i m_i f(\rho_i)$, so if $f(\rho_i) \in \mathrm{Div}^+_{d_i}(\mathbb{G})$ for all i then $f(\rho) \in \mathrm{Div}^+_d(\mathbb{G})$. Using this, we see that $X_{\mathrm{Ch}}(G)$ is the scheme of Λ-ring maps $R(G)/I \to \mathrm{Div}(\mathbb{G})$ such that $f(\rho_i) \in \mathrm{Div}^+_{d_i}(\mathbb{G})$ for $i = 1, \dots, t$.

We next give two results that help us to understand $R(G)/I$ without computing $R(G)$.

Proposition 9.5. *Let h be the number of conjugacy classes of elements $g \in G$ whose order is a power of p. Then $R(G)/I \simeq \mathbb{Z}^h$ as Abelian groups.*

Proof. We already know that $R(G)/I$ is a free Abelian group, so we just need to determine its rank, so it is enough to show that $\mathbb{C} \otimes R(G)/I \simeq \mathbb{C}^h$. Let C be the set of conjugacy classes of p-power order, and let C' be the set of all other conjugacy classes. Let $U(G)$ be the ideal of virtual representations $V \in R(G)$ whose character is zero on C'. It is well-known that $\mathbb{C} \otimes R(G) \simeq F(C \amalg C', \mathbb{C})$. This isomorphism carries $U(G)$ to $F(C, \mathbb{C})$ and I to $F(C', \mathbb{C})$ so the map $\mathbb{C} \otimes U(G) \to \mathbb{C} \otimes R(G)/I$ is an isomorphism. Clearly $\dim_{\mathbb{C}} \mathbb{C} \otimes U(G) = |C| = h$, and the claim follows. □

Remark 9.6. The proof shows that $U(G)$ is a subgroup of finite index in $R(G)/I$. This index need neither be a power of p nor coprime to p, as one sees by taking $G = \Sigma_3$ and $p = 2$ or 3; the index is 2 in both cases.

Proposition 9.7. *There is a natural isomorphism $\mathbb{Z}_p \otimes R(G)/I = KU^0_p(BG)$ (where KU_p is the p-adic completion of the complex K-theory spectrum.)*

Proof. Let KU_G be the usual G-spectrum for equivariant K-theory, and let L_G be its p-completion. It is well-known that $KU^0_G(S^0) = R(G)$, which is free Abelian of finite rank, and it follows that $L^0_G(S^0) = R(G)^\wedge_p = R(G) \otimes \mathbb{Z}_p$. We give this ring and all its quotients the p-adic topology, or equivalently the profinite topology, which is compact. The argument of the Atiyah-Segal completion theorem shows that $KU^0_p BG = L^0 EG$ is the completion of $R(G)^\wedge_p$ at the augmentation ideal J_G. By a compactness argument, we deduce that the map $R(G)^\wedge_p \to KU^0_p BG$ is surjective; the kernel is $J^\infty_G := \bigcap_k J^k_G$. Now let P be a Sylow p-subgroup, so by the same arguments $KU^0_p BP = R(P)^\wedge_p / J^\infty_P$. It is well-known that $J^N_P \leq pR(P)^\wedge_p$ for $N \geq 0$ (use the fact that $x^p - \psi^p(x) \in pR(P)$ for all $x \in R(P)$, for example) and it follows that $J^\infty_P = 0$, so $KU^p_0 BP = R(P)^\wedge_p$. We now have a diagram as follows.

$$\begin{tikzcd}
 & J^\infty_G \arrow[r] \arrow[d, tail] & J^\infty_P = 0 \arrow[d] \\
I^\wedge_p \arrow[r, tail] & R(G)^\wedge_p \arrow[r] \arrow[d, two heads] & R(P)^\wedge_p \arrow[d, "\simeq"] \\
 & KU^0_p BG \arrow[r, tail] & KU^0_p BP.
\end{tikzcd}$$

We have seen that the columns are short exact. As $\mathbb{Z}_p$ is flat over $\mathbb{Z}$ and I, $R(G)$ and $R(P)$ are finitely generated Abelian groups, we see that the middle row is left exact. The bottom horizontal map is injective by a transfer argument. By a diagram chase we deduce that $I^\wedge_p = J^\infty_G$, so $KU^0_p BG = R(G)^\wedge_p / I^\wedge_p = (R(G)/I) \otimes \mathbb{Z}_p$ as claimed. □

10. Finiteness

It is a fundamental fact that the scheme $X(G)$ is finite over X, or equivalently that E^0BG is a finitely generated module over E^0. This is proved in the present generality as [4, Corollary 4.4]; the argument is the same as in [5]. In this section we show that $X_{\mathrm{Ch}}(G)$ is also finite over X. We also study some auxiliary schemes that come up in the proof, as they turn out to be useful in specific computations.

Definition 10.1. For any $v \geq 0$ we put

$$\mathbb{G}(v) = \ker(p^v \colon \mathbb{G} \to \mathbb{G}).$$

In terms of our coordinate, we have $\mathbb{G}(v) = \mathrm{spf}(E^0[\![x]\!]/[p^v](x))$. Next, recall that $\mathbb{G}(v) \leq \mathbb{G}$ is a divisor of degree p^{nv}, where n is the height of $\mathbb{G}$. For any $m \geq 0$ we let $\mathbb{G}(v,m)$ be m times this divisor, considered as a subscheme of $\mathbb{G}$; in other words $\mathbb{G}(v,m) = \mathrm{spf}(E^0[\![x]\!]/[p^v](x)^m)$. For any formal scheme Y over X such that $\mathcal{O}_Y$ is a free module of finite rank r over $\mathcal{O}_X$, we define Y^d/Σ_d to be spf of the d'th symmetric tensor power of $\mathcal{O}_Y$ over $\mathcal{O}_X$. If $\{e_1, \dots, e_r\}$ is a basis for $\mathcal{O}_Y$ over $\mathcal{O}_X$ then the monomials $e^\alpha := \prod_{i=1}^r e_i^{\alpha_i}$ with $\sum_i \alpha_i = d$ form a basis for $\mathcal{O}_{Y^d/\Sigma_d}$, so this ring is free of rank $\binom{r+d-1}{d}$ over $\mathcal{O}_X$. We will use this construction in the cases $Y = \mathbb{G}(v)$ and $Y = \mathbb{G}(v,m)$. Finally, we define $Z(v,d)$ to be the scheme of divisors $D \in \mathrm{Div}_d^+(\mathbb{G})$ such that $\psi^{p^v} D = d[0]$.

Theorem 10.2. *The scheme $Z(d,v)$ is finite and flat over X, of degree p^{ndv}. There are closed inclusions*

$$\mathbb{G}(v)^d/\Sigma_d \to Z(v,d) \to \mathbb{G}(v,d)^d/\Sigma_d \to \mathrm{Div}_d^+(\mathbb{G}).$$

The first two of these are infinitesimal thickenings, in other words the corresponding maps of rings are surjective with nilpotent kernel. If $\mathcal{O}_X$ is a field (necessarily of characteristic p) then $Z(v,d)$ is the fibre of the nv-fold relative Frobenius map

$$F^{nv}_{\mathrm{Div}_d^+(\mathbb{G})/X} \colon \mathrm{Div}_d^+(\mathbb{G}) \to \mathrm{Div}_d^+((F_X^{nv})^*\mathbb{G}),$$

so

$$\mathcal{O}_{Z(v,d)} = \mathcal{O}_X[\![c_1, \dots, c_d]\!]/(c_i^{p^{nv}}).$$

Proof. First suppose that $\mathcal{O}_X$ is a complete regular local ring. (In the topological context, this occurs when E is Landweber exact.) Consider the following diagram:

$$\begin{tikzcd}
Z(v,d) \arrow[r, tail] \arrow[d] & \mathrm{Div}_d^+(\mathbb{G}) \arrow[d, two heads, "\psi^{p^v}"'] & \mathbb{G}^d \arrow[l, two heads, "\pi"'] \arrow[d, two heads, "p^v"] \\
X \arrow[r, "\zeta"'] & \mathrm{Div}_d^+(\mathbb{G}) & \mathbb{G}^d \arrow[l, two heads, "\pi"]
\end{tikzcd}$$

In the right hand square, all the corresponding rings are complete regular local rings. A finite injective map of such rings always makes the target into a free module over the source [1, 2.2.7 and 2.2.11]. The maps π^* and $(p^v)^*$ are finite injective maps of degrees $d!$ and p^{ndv}. It follows that $(\psi^{p^v})^*$ is finite and injective, and thus (as $\deg(fg) = \deg(f)\deg(g)$ in this context) that ψ^{p^v} is flat of degree p^{ndv}. The left hand square is a pullback by definition, and it follows that $Z(v,d)$ is flat of degree p^{ndv} over X. Using [4, Proposition 5.2], it is not hard to deduce that this result remains true even if $\mathcal{O}_X$ is not regular.

Next, let x be a coordinate on $\mathbb{G}$. Then $\{x^i \mid i < p^{nv}\}$ is a basis for $\mathcal{O}_{\mathbb{G}(v)}$ over $\mathcal{O}_X$, and $\{x^i \mid i < p^{nmv}\}$ is a basis for $\mathcal{O}_{\mathbb{G}(v,m)}$. Using this we obtain bases for the rings

$$A = \mathcal{O}_{\mathbb{G}(v)^d} = \mathcal{O}_X[\![x_1, \dots, x_d]\!]/([p^v](x_i))$$

and

$$A' = \mathcal{O}_{\mathbb{G}(v,m)^d} = \mathcal{O}_X[\![x_1, \dots, x_d]\!]/([p^v](x_i)^m)$$

that are permuted by Σ_d, and the orbit sums give bases for the rings

$$B = \mathcal{O}_{\mathbb{G}(v)^d/\Sigma_d} = A^{\Sigma_d}$$

and

$$B' = \mathcal{O}_{\mathbb{G}(v,m)/\Sigma_d} = (A')^{\Sigma_d}.$$

Using these, it is easy to see that the map $B' \to B$ is surjective, so the map $\mathbb{G}(v)^d/\Sigma_d \to \mathbb{G}(v,m)^d/\Sigma_d$ is a closed inclusion. A similar argument shows that $\mathbb{G}(v,m)^d/\Sigma_d$ is a closed subscheme of $\mathrm{Div}_d^+(\mathbb{G})$.

Next, put

$$J = \ker(A' \to A) = ([p^v](x_1), \dots, [p^v](x_d)).$$

This is clearly a nilpotent ideal, and $\ker(B' \to B) = B' \cap J$ which is a nilpotent ideal in B'. Thus our map $\mathbb{G}(v)^d/\Sigma_d \to \mathbb{G}(v,m)^d/\Sigma_d$ is an infinitesimal thickening.

It is clear that $\mathbb{G}(v)^d/\Sigma_d$ is contained in $Z(v,d)$. Next, let $W(v,d)$ be the preimage of $Z(v,d)$ in $\mathbb{G}^d$, or equivalently the scheme of d-tuples $\underline{a} = (a_1, \dots, a_d) \in \mathbb{G}^d$ such that $\sum_i [p^v a_i] = d[0]$. If $\underline{a} \in W(v,d)$ then for each i we have $p^v a_i \in d[0]$ so $a_i \in d.(p^v)^{-1}[0] = d.\mathbb{G}(v) = \mathbb{G}(v,d)$. This means that $\underline{a} \in \mathbb{G}(v,d)^d$ and thus $\sum_i [a_i] \in \mathbb{G}(v,d)^d/\Sigma_d$, so the map $W(v,d) \xrightarrow{\pi} Z(d,v) \to \mathrm{Div}_d(\mathbb{G})$ factors through $\mathbb{G}(v,d)^d/\Sigma_d$. As π is faithfully flat, it follows that $Z(v,d) \subseteq \mathbb{G}(v,d)^d/\Sigma_d$ as claimed.

We now have maps

$$\mathbb{G}(v)^d/\Sigma_d \xrightarrow{i} Z(v,d) \xrightarrow{j} \mathbb{G}(v,d)^d/\Sigma_d \xrightarrow{k} \mathrm{Div}_d^+(\mathbb{G}).$$

We know that ji, k and kj are closed inclusions and that ji is an infinitesimal thickening. It follows easily that i, j and k are closed inclusions and i and j are infinitesimal thickenings.

Now suppose that X is a field of characteristic p. We then have an iterated Frobenius map $F_X^{nv} \colon X \to X$ corresponding to the ring map $a \mapsto a^{p^{nv}}$ and thus a formal group $\mathbb{G}' = (F_X^{nv})^*\mathbb{G}$ over X. The map $F_{\mathbb{G}}^{nv}$ gives rise to a map $f = F_{\mathbb{G}/X}^{nv} \colon \mathbb{G} \to \mathbb{G}'$. As $\mathbb{G}$ has height n, the map $p^v \colon \mathbb{G} \to \mathbb{G}$ factors as $\mathbb{G} \xrightarrow{f} \mathbb{G}' \xrightarrow{g} \mathbb{G}$, where g is an isomorphism. This is just the geometric statement of the fact that $[p^v](x) = \gamma(x^{p^{nv}})$ for some invertible power series γ. By definition $Z(v,d)$ is the fibre of the map $\mathrm{Div}_d^+(\mathbb{G}) \to \mathrm{Div}_d^+(\mathbb{G})$ induced by $p^v \colon \mathbb{G} \to \mathbb{G}$, and it follows easily that it is also the fibre of the map $\mathrm{Div}_d^+(\mathbb{G}) \to \mathrm{Div}_d^+(\mathbb{G}')$ induced by f. It is easy to identify this with the map $F^{nv}_{\mathrm{Div}_d^+(\mathbb{G})}$. If we use the usual generators for the coordinate rings of $\mathrm{Div}_d^+(\mathbb{G})$ and $\mathrm{Div}_d^+(\mathbb{G}')$ then the corresponding ring map sends c_k to $c_k^{p^{nv}}$, so $\mathcal{O}_{Z(v,d)} = \mathcal{O}_X[\![c_1, \dots, c_d]\!]/(c_k^{p^{nv}})$. ∎

Corollary 10.3. *The scheme $X_{Ch}(G)$ is finite over X.*

Proof. Let $V_1, \dots, V_h$ be the irreducible representations of G, and let $d_1, \dots, d_h$ be their degrees. As in the proof of Proposition 5.3, we see that $X_{\mathrm{Ch}}(G)$ is a closed subscheme of $\prod_i \mathrm{Div}_{d_i}^+(\mathbb{G})$. Now let p^v be the p-part of the exponent of G. We see from Lemma 4.3 that $X_{\mathrm{Ch}}(G)$ is actually contained in $\prod_i Z(v, d_i)$, which is finite over X by the theorem. It follows that $X_{\mathrm{Ch}}(G)$ itself is finite, as claimed. ∎

Corollary 10.4. *For any divisor $D \in \mathrm{Div}(\mathbb{G})(A)$ there exist $w \geq 0$ such that $\psi^k(D) = \psi^l(D)$ whenever $k, l \in \mathbb{Z}_p$ with $k = l \pmod{p^w}$.*

Proof. We can reduce easily to the case where $D \in \mathrm{Div}_d^+(\mathbb{G})$ for some $d \geq 0$. Lemma 4.3 tells us that $\psi^{p^v} D = d[0]$ for some v, so $D \in Z(v,d)(A) \leq \mathbb{G}(v,d)^d/\Sigma_d(A)$. Next note that A is a discrete $\mathcal{O}_X$-algebra so p is nilpotent in A, say $p^r = 0$. It follows that $[p^r](x) = f(x^p)$ for some power series f with $f(0) = 0$, and thus that $[p^{rs}](x)$ is divisible by x^{p^s}. Thus, for large u we have $[p^u](x) = 0 \pmod{x^d}$, so $[p^{u+v}](x) = 0 \pmod{[p^v](x)^d}$, so $\mathbb{G}(v,d) \leq \mathbb{G}(u+v)$. If we put $w = u+v$ this tells us that $D \in \mathbb{G}(w)^d/\Sigma_d$, and the action of ψ^k on $\mathbb{G}(w)^d/\Sigma_d$ clearly depends only on the congruence class of k mod p^w, as required. ∎

11. Generalised character theory

In [5], Hopkins, Kuhn and Ravenel describe $\mathbb{Q} \otimes E^0BG$ in terms of "generalised characters". In this section we will give an analogous but less precise description of $\mathbb{Q} \otimes C(E,G)$.

To explain the HKR theory, write $\Theta = (\mathbb{Q}_p/\mathbb{Z}_p)^n$ and $\Theta^* = \mathrm{Hom}(\Theta, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Z}_p^n$. (Elsewhere these are denoted by Λ and Λ^*, but there are enough Λ's in this paper already.) Let $\Theta(v)$ be the subgroup of Θ killed by p^v, and let $\mathrm{Level}(v, \mathbb{G})$ be the scheme of maps $\phi \colon \Theta(v) \to \mathbb{G}$ such that $\sum_{a \in \Theta(v)} [\phi(a)] \leq \mathbb{G}(v)$ in $\mathrm{Div}^+(\mathbb{G})$. See [9] for more information about these schemes. Put $D_m = \mathcal{O}_{\mathrm{Level}(m,\mathbb{G})}$ and $D = \varinjlim_m D_m$ and $L = \mathbb{Q} \otimes D$. This is a free module of countable rank over $\mathbb{Q} \otimes \mathcal{O}_X$. If $\mathbb{G}$ is a universal deformation (as in the case considered by HKR) then it can be described more explicitly: the Weierstrass preparation theorem

implies that $[p^v](x)$ is a unit multiple of a monic polynomial $g_v(x)$ of degree p^{nv}, and L is obtained from $\mathbb{Q}\otimes\mathcal{O}_X$ by adjoining full set of roots for $g_v(x)$ for all v.

Now let $\Omega(G)$ be the set of G-conjugacy classes of homomorphisms $\Theta^*\to G$, and let $F(\Omega(G),L)$ be the ring of all functions $u\colon\Omega(G)\to L$ (with pointwise operations). HKR construct an isomorphism

$$\tau\colon L\otimes_{\mathcal{O}_X}E^0BG\to F(\Omega(G),L).$$

They work with a particular admissible cohomology theory E, but it is not hard to extend their result to all admissible theories; see [4, Proposition 5.2 and Appendix B] for some pointers.

Now consider the Λ-semiring $\mathbb{N}[\Theta]=\coprod_d\Theta^d/\Sigma_d$ and the Λ-ring $\mathbb{Z}[\Theta]$. If we give Θ^* its p-adic topology then every subgroup of finite index is open and any continuous homomorphism $\Theta^*\to GL_n(\mathbb{C})$ factors through a finite quotient of Θ^*. Using this we can identify $\mathbb{N}[\Theta]$ with the semiring of continuous representations of Θ^*, and $\mathbb{Z}[\Theta]$ with the corresponding ring of virtual representations.

Definition 11.1. We say that a Λ-ring homomorphism $f\colon R(G)\to\mathbb{Z}[\Theta]$ is *positive* if $f(R^+(G))\subseteq\mathbb{Z}[\Theta]^+$. We write $\Omega_{\mathrm{Ch}}(G)$ for the set of Λ-semiring homomorphisms $R^+(G)\to\mathbb{Z}[\Theta]^+$, or equivalently the set of positive Λ-ring homomorphisms $R(G)\to\mathbb{Z}[\Theta]$.

Remark 11.2. The arguments of Lemma 4.3, Corollary 5.5 and Proposition 9.1 show that any positive homomorphism $f\colon R(G)\to\mathbb{Z}[\Theta]$ of Λ-rings automatically sends $R^+_d(G)$ to $\mathbb{Z}[\Theta]^+_d$ and I to 0 (where I is the kernel of the restriction map to a Sylow subgroup). There is also an evident analogue of Remark 9.4 for $\Omega_{\mathrm{Ch}}(G)$.

Remark 11.3. From the definitions we know that the Λ-operations determine the Adams operations. Conversely, it is well-known and easy to check that the Adams operations determine the Λ-operations rationally. As $\mathbb{Z}[\Theta]$ is torsion-free, it follows that a ring map $f\colon R(G)\to\mathbb{Z}[\Theta]$ preserves the Λ-operations iff it preserves the Adams operations. The corresponding statement for homomorphisms $R(G)\to\mathrm{Div}(\mathbb{G})$ is false, however.

Theorem 11.4. *There are natural maps $\kappa\colon\Omega(G)\to\Omega_{Ch}(G)$ and $\tau_{Ch}\colon L\otimes C(E,G)\to L\otimes E^0BG$ making the following diagram commute:*

$$\begin{tikzcd}
L\otimes C(E,G) \arrow[r, "1\otimes\theta^*"] \arrow[d, "\tau_{Ch}"'] & L\otimes E^0BG \arrow[d, "\tau"] \\
F(\Omega_{Ch}(G),L) \arrow[r, "\kappa^*"'] & F(\Omega(G),L).
\end{tikzcd}$$

(Here the tensor products are taken over E^0.) Moreover, the map τ_{Ch} is surjective with nilpotent kernel.

Proof. For brevity we will write $C(L,G)=L\otimes C(E,G)$ and $L^0BG=L\otimes E^0BG$. This is a slight abuse because these functors do not arise from a spectrum L. We also let v be any integer greater than or equal to the p-adic valuation of the exponent of G.

Any homomorphism $u\colon\Theta^*\to G$ factors through $\Theta^*/p^v=\Theta(v)^*$ and thus is automatically continuous (for the discrete topology on G). It thus gives a positive homomorphism $u^*\colon R(G)\to\mathbb{Z}[\Theta]$, and it is well-known that this depends only on the conjugacy class of u, so this construction gives a natural map $\kappa\colon\Omega(G)\to\Omega_{\mathrm{Ch}}(G)$.

It is easy to see using Adams operations that any positive homomorphism $f\colon R(G)\to\mathbb{Z}[\Theta]$ actually lands in the subring $\mathbb{Z}[\Theta(v)]$. Suppose we have a level structure $\phi\colon\Theta(v)\to\mathbb{G}(A)$. As $\Theta(v)$ is a finite Abelian group, this gives rise as in Theorem 7.1 to a positive homomorphism $R(\Theta^*/p^v)=\mathbb{Z}[\Theta(v)]\to\mathrm{Div}(\mathbb{G})(A)$, which we can compose with f to get a positive homomorphism $R(G)\to\mathrm{Div}(\mathbb{G})(A)$, or in other words a point of $X_{\mathrm{Ch}}(G)(A)$, which we call $\rho_{\mathrm{Ch}}(f,\phi)$. This construction produces a map $\rho_{\mathrm{Ch}}\colon\Omega_{\mathrm{Ch}}(G)\times\mathrm{Level}(v,\mathbb{G})\to X_{\mathrm{Ch}}(G)$ of formal schemes over X, corresponding to a map $\rho^*_{\mathrm{Ch}}\colon C(E,G)\to F(\Omega_{\mathrm{Ch}}(G),D_v)\subset F(\Omega_{\mathrm{Ch}}(G),L)$. After tensoring by L we obtain the required map $\tau_{\mathrm{Ch}}\colon C(L,G)\to F(\Omega_{\mathrm{Ch}}(G),L)$.

We next recall the definition of τ. Suppose that $u\colon\Theta^*/p^v\to G$ and $\phi\in\mathrm{Level}(v,\mathbb{G})\subset\mathrm{Hom}(\Theta(v),\mathbb{G})=X(\Theta^*/p^v)$. We then have a point $\rho(u,\phi):=X(u)(\phi)\in X(G)$. This construction gives a map $\rho\colon\Omega(G)\times\mathrm{Level}(v,\mathbb{G})\to X(G)$ and thus a map $\rho^*\colon E^0BG\to F(\Omega(G),D_v)\subset F(\Omega(G),L)$. After tensoring by L we obtain the required map τ.

One can check from the definitions that the following diagram commutes:

$$\begin{tikzcd}
X_{\mathrm{Ch}}(G) & & X(G) \arrow[ll, "\theta"'] \\
\Omega_{\mathrm{Ch}}(G) \times \mathrm{Level}(m, \mathbb{G}) \arrow[u, "\rho_{\mathrm{Ch}}"] & & \Omega(G) \times \mathrm{Level}(m, \mathbb{G}). \arrow[ll, "\kappa \times 1"] \arrow[u, "\rho"']
\end{tikzcd}$$

It follows easily that the diagram in the statement of the theorem commutes.

To understand τ_{Ch} more explicitly, let $\phi \in \mathrm{Level}(v, \mathbb{G})(D_v)$ be the universal example of a level structure. For any element $a \in \Theta(v)$ we then have a point $\phi(a) \in \mathbb{G}(D_v)$ and thus an element $x_a := x(\phi(a)) \in D_v$. These elements satisfy $x_{a+b} = x_a +_F x_b$ and $\mathbb{G}(v) = \sum_{a \in \Theta(v)} [\phi(a)]$ as divisors, or equivalently $[p^v](t)$ is a unit multiple of $\prod_a (t - x_a)$ in $D_v[\![t]\!]$. It is also known that $x_a - x_b$ is invertible in L whenever $a \neq b$ (because it is a unit multiple of x_{a-b}, which divides $\prod_{c \neq 0} x_c = \pm [p^v]'(0) = \pm p^v$). Let the representations V_i and the elements $c_{ik} \in C(E, G)$ be as in the proof of Proposition 5.3. If $f \in \Omega_{\mathrm{Ch}}(G)$ and $f(V_i) = [a_1] + \ldots + [a_d] \in \mathbb{N}[\Theta(v)]$ then $\tau_{\mathrm{Ch}}(c_{ik})(f)$ is the k'th symmetric function in the variables $x_{a_1}, \ldots, x_{a_d}$, and this characterises τ_{Ch}.

We next show that τ_{Ch} is surjective. For any $u \in \Omega_{\mathrm{Ch}}(G)$ we define $\epsilon_u \colon C(L, G) \to L$ by $\epsilon_u(c) = \tau_{\mathrm{Ch}}(c)(u)$, and we put $I_u = \ker(\epsilon_u) \leq C(L, G)$. By the Chinese Remainder Theorem, it will suffice to show that $I_u + I_v = C(L, G)$ whenever $u \neq v$. If $u \neq v$ we can choose $V \in R^+(G)$ such that $u(V) \neq v(V) \in \mathbb{N}[\Theta]$. If $u(V) = \sum_a m_a[a]$ and $v(V) = \sum_a n_a[a]$ then we must have $m_b \neq n_b$ for some b, and without loss we may assume $m_b > n_b$. Define

$$\begin{aligned}
k_a &= \min(n_a, m_a) \\
C &= \sum_a k_a[a] \\
A &= \sum_a (n_a - k_a)[a] \\
B &= \sum_a (m_a - k_a)[a].
\end{aligned}$$

We can write A in the form $[a_1] + \ldots + [a_d]$ with $a_i \neq b$ for all i. We also write $f_A(t) = \prod_i (t - x_{a_i})$, so $f_A(x_b)$ is invertible in L. On the other hand, the representation $V \in R^+_d(G)$ gives rise in a tautological way to a divisor $D_V \in \mathrm{Div}^+_d(\mathbb{G})(C(E, G))$ with equation $f_{D_V}(t) \in C(E, G)[t]$, say. We have $\epsilon_u f_{D_V}(t) = f_{A+C}(t) = f_A(t) f_C(t)$ and $\epsilon_v f_{D_V}(t) = f_B(t) f_C(t)$. The polynomial $f_C(t)$ is monic and thus is not a zero-divisor, so $f_A(t) = f_B(t) \pmod{I_u + I_v}$. We evidently have $f_B(x_b) = 0$ so $f_A(x_b) = 0 \pmod{I_u + I_v}$. As $f_A(x_b)$ is invertible in L, we deduce that $I_u + I_v = 1$ as required.

Finally, we must show that the kernel of τ_{Ch} is nilpotent. This kernel is the intersection of the ideals I_u, so by well-known arguments it suffices to show that every prime ideal in $C(L, G)$ contains I_u for some u. To see this, put $R = C(L, G)$ and let $\mathfrak{p} \leq R$ be a prime ideal. If $D \in \mathrm{Div}^+_d(\mathbb{G})(C(E, G))$ is a divisor satisfying $\psi^{p^v}(D) = d[0]$, then Theorem 10.2 implies that $D \in \mathbb{G}(v, d)^d / \Sigma_d$ and thus that $D \leq d^2.\mathbb{G}(v)$ as divisors, or equivalently $f_D(t)$ divides $[p^v](t)^{d^2}$, which is a unit multiple in $D_v[t]$ of $\prod_{a \in \Theta(v)} (t - x_a)^{p^v}$. Now let K be the field of fractions of $R/\mathfrak{p}$, and note that $x_a - x_b$ is invertible in L and thus in K when $a \neq b$. As $K[t]$ is a unique factorisation domain, we see that $f_D(t) = \prod_a (t - x_a)^{m_a}$ in $K[t]$ for a unique system of integers m_a. We define $w(D) = \sum_a m_a[a] \in \mathbb{Z}[\Theta(v)]^+_d$. In particular, if $V \in R^+_d(G)$ we can let D_V be the tautologically associated divisor over $C(E, G)$ and put $u(V) = w(D_V)$. One can check that this gives a homomorphism $u \colon R^+(G) \to \mathbb{Z}[\Theta]$ of Λ-semirings, or in other words an element $u \in \Omega_{\mathrm{Ch}}(G)$. From the construction it is automatic that $I_u \leq \mathfrak{p}$. $\square$

Example 11.5. If G is Abelian, it is easy to see that $\Omega_{\mathrm{Ch}}(G) = \mathrm{Hom}(G^*, \Theta) \simeq \mathrm{Hom}(\Theta^*, G) = \Omega(G)$ and that κ is an isomorphism.

Example 11.6. Consider the symmetric group $G = \Sigma_k$. This acts in an obvious way on $\mathbb{C}^k$, and we call this representation π. It is known that π generates $R(G)$ as a Λ-ring. Thus, an element $f \in \Omega_{\mathrm{Ch}}(G)$ is determined by the value $f(\pi) \in \mathbb{Z}[\Theta]^+_k$.

As discussed in [12], the set $\Omega(G)$ can be identified with the set of isomorphism classes of sets of order k with an action of Θ^*. For any finite subgroup $A<\Theta$ we have a homomorphism $\Theta^*\to A^*$ and thus an action of Θ^* on A^*. Note that 0^* is just a single point with trivial action. We write $m.A^*$ for the disjoint union of m copies of A^*. If T is a finite Θ^*-set, then T can be written in an essentially unique way in the form $\coprod_i m_i.A_i^*$.

If we write $[A]=\sum_{a\in A}[a]\in\mathbb{Z}[\Theta]$ then by working through the definitions we find that $\kappa(\coprod_i m_i.A_i^*)(\pi)=\sum_i m_i[A_i]$, which effectively determines κ.

It is now easy to exhibit cases in which κ is not injective. For example, suppose that $p=2$ and $n>1$ and $k=6$. We can then find two distinct, nonzero elements $a,b\in\Theta(1)$ and put $c=a+b$. Let A, B and C be the groups generated by a, b and c respectively, and put $V=A+B=\{0,a,b,c\}$. Then

$$\kappa(V^*\amalg 2.0^*)(\pi)=\kappa(A^*\amalg B^*\amalg C^*)(\pi)=3[0]+[a]+[b]+[c],$$

so κ is not injective. In Section 15 we will give examples where κ is not surjective.

12. Calculating $\Omega_{\mathrm{CH}}(G)$

In this section we define sets $\Omega'(G)$ and $\Omega''(G)$ which in some cases may be easier to compute than $\Omega(G)$ or $\Omega_{\mathrm{Ch}}(G)$, and we define natural maps

$$\Omega(G)\twoheadrightarrow\Omega'(G)\rightarrowtail\Omega_{\mathrm{Ch}}(G)\rightarrowtail\Omega''(G).$$

Definition 12.1. Let C be the set of conjugacy classes of elements of p-power order in G. We let the multiplicative monoid $\mathbb{Z}$ act on Θ^* in the obvious way, and on C by $k.[g]=[g^k]$. We say that two homomorphisms $u,v\colon\Theta^*\to G$ are pointwise conjugate if $u(a)$ is conjugate to $v(a)$ for all $a\in\Theta^*$. We recall the definitions of $\Omega(G)$ and $\Omega_{\mathrm{Ch}}(G)$ and define new sets $\Omega'(G)$ and $\Omega''(G)$ as follows:

$$\begin{aligned}\Omega(G)&=\mathrm{Hom}(\Theta^*,G)/\text{conjugacy}\\ \Omega'(G)&=\mathrm{Hom}(\Theta^*,G)/\text{pointwise conjugacy}\\ \Omega''(G)&=\{\mathbb{Z}\text{-equivariant continuous maps }\Theta^*\to C\}\\ \Omega_{\mathrm{Ch}}(G)&=\{\text{ positive }\Lambda\text{-ring homomorphisms }R(G)\to\mathbb{Z}[\Theta]\}.\end{aligned}$$

Proposition 12.2. *There are natural maps as follows:*

$$\Omega(G)\twoheadrightarrow\Omega'(G)\rightarrowtail\Omega_{Ch}(G)\rightarrowtail\Omega''(G).$$

Proof. There are evident natural maps

$$\Omega(G)\twoheadrightarrow\Omega'(G)\rightarrowtail\Omega''(G).$$

We have also already constructed a map $\kappa\colon\Omega(G)\to\Omega_{\mathrm{Ch}}(G)$. If $u,v\colon\Theta^*\to G$ are pointwise-conjugate then the induced maps from class functions on G to class functions on Θ^* are evidently the same, so the induced maps $R(G)\to\mathbb{Z}[\Theta]$ are the same, so $\kappa(u)=\kappa(v)$. This shows that κ factors through the projection $\Omega(G)\to\Omega'(G)$.

We next define a map $\xi\colon\Omega_{\mathrm{Ch}}(G)\to\Omega''(G)$. Suppose that $u\in\Omega_{\mathrm{Ch}}(G)$ and $a\in\Theta^*=\mathrm{Hom}(\Theta,S^1)$. Then a extends in a natural way to give a $\mathbb{C}$-algebra map $\hat{a}\colon\mathbb{C}[\Theta]\to\mathbb{C}$ and thus a ring map $(1_{\mathbb{C}}\otimes u)\hat{a}\colon\mathbb{C}\otimes R(G)\to\mathbb{C}$. Using the fact that $\mathbb{C}\otimes R(G)$ is the set of $\mathbb{C}$-valued class functions on G, we see that $\mathrm{Hom}_{\mathbb{C}-\mathrm{Alg}}(\mathbb{C}\otimes R(G),\mathbb{C})$ can be identified with the set of conjugacy classes in G. Thus there exists $h\in G$ (unique up to conjugation) such that $(1\otimes u)(\hat{a}(V))=\chi_V(h)$ for all $V\in R(G)$. We can choose m so that $u(V)\in\mathbb{Z}[\Theta(m)]$ for all V, and then we have $\chi_V(h^{p^m})=\chi_{\psi^{p^m}V}(h)=\chi_{\dim(V)}(h)=\dim(V)$ for all V, so $h^{p^m}=1$. This means that the conjugacy class $[h]$ lies in C, so we can define $\xi(u)(a)=[h]$. We leave it to the reader to check that this gives a map $\xi\colon\Omega_{\mathrm{Ch}}(G)\to\Omega''(G)$ as claimed. The maps $\hat{a}\colon\mathbb{Z}[\Theta]\to\mathbb{C}$ (as a runs over Θ^*) are jointly injective, an it follows that ξ is injective. One can also check that the composite $\Omega'(G)\to\Omega_{\mathrm{Ch}}(G)\to\Omega''(G)$ is just the obvious inclusion, which implies that the map $\Omega'(G)\to\Omega_{\mathrm{Ch}}(G)$ is injective. $\square$

13. Special divisors

In this section we study "special" divisors, which are related to the special unitary group in the same way that arbitrary divisors are related to the full unitary group.

Definition 13.1. A divisor $D \in \mathrm{Div}_d^+(\mathbb{G})$ is *special* if $\lambda^d D = [0]$. We write $\mathrm{SDiv}_d^+(\mathbb{G})$ for the scheme of special divisors.

Proposition 13.2. *We have $\mathcal{O}_{\mathrm{SDiv}_d^+(\mathbb{G})} = \mathcal{O}_X[\![c_2, \ldots, c_d]\!]$. In the topological situation this can be identified with $E^0BSU(d)$.*

Proof. Put $A = \mathcal{O}_{\mathbb{G}^d} = \mathcal{O}_X[\![x_1, \ldots, x_d]\!]$ and $A' = \mathcal{O}_{\mathrm{Div}_d^+(\mathbb{G})} = A^{\Sigma_d} = \mathcal{O}_X[\![c_1, \ldots, c_d]\!]$. Here c_i is the i'th elementary symmetric function, and in particular $c_1 = \sum_i x_i$. Put $c_1' = \sum_i^F x_i \in A'$. If we regard λ^d as a map $\mathrm{Div}_d^+(\mathbb{G}) \to \mathrm{Div}_1^+(\mathbb{G}) = \mathbb{G}$ then $c_1' = x \circ \lambda^d$, so we see that $\mathcal{O}_{\mathbb{H}_3} = A/c_1'$ and $\mathcal{O}_{\mathrm{SDiv}_d^+(\mathbb{G})} = A'/c_1'$. Next, observe that the inclusion $A' \to A$ induces an inclusion $A'/(c_1^2, c_2, \ldots, c_d) \to A/(x_1, \ldots, x_d)^2$. We have $c_1' = c_1 \pmod{(x_1, \ldots, x_d)^2}$ so $c_1' = c_1 \pmod{c_1^2, c_2, \ldots, c_d}$. It follows easily that $A' = \mathcal{O}_X[\![c_1', c_2, \ldots, c_d]\!]$ and thus that $A'/c_1' = \mathcal{O}_X[\![c_2, \ldots, c_d]\!]$. □

We next put $\mathbb{H}_d = \ker(\mathbb{G}^d \xrightarrow{+} \mathbb{G})$. If we let $q\colon \mathbb{G}^d \to \mathrm{Div}_d^+(\mathbb{G})$ be the usual projection (which is finite and faithfully flat, with degree $d!$) then $\mathbb{H}_d = q^{-1}\,\mathrm{SDiv}_d^+(\mathbb{G})$. It follows that the map $q\colon \mathbb{H}_d \to \mathrm{SDiv}_d^+(\mathbb{G})$ is also finite and faithfully flat, with the same degree. It clearly factors through $\mathbb{H}_d/\Sigma_d := \mathrm{spf}(\mathcal{O}_{\mathbb{H}_d}^{\Sigma_d})$, and one would like the induced map $q\colon \mathbb{H}_d/\Sigma_d \to \mathrm{SDiv}_d^+(\mathbb{G})$ to be an isomorphism. However, quotient constructions in algebraic geometry are never as simple as one would like, and we do not know whether this is true in general; certainly it becomes false if we remove our assumption that $\mathbb{G}$ has finite height. For example, consider the case where $\mathbb{G}$ is the additive group over $\mathbb{F}_2$ and $d = 2$; then $2a = 0$ for all $a \in \mathbb{G}$ so Σ_2 acts trivially on $\mathbb{H}_2 = \{(a, -a) \mid a \in \mathbb{G}\}$ so the map $q\colon \mathbb{H}_d/\Sigma_d \to \mathrm{SDiv}_d^+(\mathbb{G})$ has degree two. However, we do have the following partial result.

Proposition 13.3. *If d is invertible in $\mathcal{O}_X$ then* $\mathrm{SDiv}_d^+(\mathbb{G}) = \mathbb{H}_d/\Sigma_d$.

Proof. As d is invertible in $\mathcal{O}_X$, multiplication by d is an automorphism of $\mathbb{G}$. Define maps $\mathbb{G}\times\mathbb{H}_d \xrightarrow{f} \mathbb{G}^d \xrightarrow{g} \mathbb{G}$ by $f(a, b_1, \ldots, b_d) = (a + b_1, \ldots, a + b_d)$ and $g(b_1, \ldots, b_n) = \sum_i b_i/d$, and then define $h\colon \mathbb{G}^d \to \mathbb{G} \times \mathbb{H}_d$ by $h(\underline{b}) = (g(\underline{b}), b_1 - g(\underline{b}), \ldots, b_d - g(\underline{b}))$. Clearly h is inverse to f, so f is an isomorphism, giving an isomorphism $\mathcal{O}_{\mathbb{G}^d} = \mathcal{O}_G \widehat{\otimes} \mathcal{O}_{\mathbb{H}_d} = \mathcal{O}_{\mathbb{H}_d}[\![x]\!]$ of rings. If we let Σ_d act trivially on $\mathbb{G}$ then everything is equivariant, so we have $\mathcal{O}_{\mathbb{G}^d}^{\Sigma_d} = \mathcal{O}_{\mathbb{H}_d}^{\Sigma_d}[\![x]\!]$, so $\mathrm{Div}_d^+(\mathbb{G}) = \mathbb{G}^d/\Sigma_d = \mathbb{G} \times (\mathbb{H}_d/\Sigma_d)$. □

14. The group Σ_4

We now consider the case where $G = \Sigma_4$ and E is the 2-periodic Morava E-theory spectrum of height 2 at the prime 2. We shall show that the map $C(E, G) \to E^0BG$ is an isomorphism. To be more explicit, we need to name some representations. Note that Σ_4 acts on $\mathbb{C}^4$ with a one-dimensional fixed subspace; we let ρ be the representation of G on the quotient space. We also write ϵ for the sign representation. We let $K = E/I_2$ denote the 2-periodic Morava K-theory spectrum.

Theorem 14.1. *Let $c_2, c_3 \in E^0B\Sigma_4$ be the Chern classes of the representation $\epsilon\rho$, and let w be the Euler class of ϵ. Then $C(E, \Sigma_4) = E^0B\Sigma_4$, and this is a free module of rank 17 over E^0, with the following monomials as a basis:*

$$\begin{array}{ccccc}
1 & c_2 & c_2^2 & c_2^3 & c_3 \\
w & wc_2 & wc_2^2 & & wc_3 \\
w^2 & w^2c_2 & w^2c_2^2 & & w^2c_3 \\
w^3 & w^3c_2 & w^3c_2^2 & & w^3c_3.
\end{array}$$

Moreover, we have

$$C(K, \Sigma_4) = K^0B\Sigma_4 = C(E, \Sigma_4)/I_2 = K^0[w, c_2, c_3]/J,$$

where J is generated by the following elements:

$$\begin{aligned}
&w^4\ ,\ c_3^2\ ,\ c_2c_3\ ,\\
&c_2^4+w^2c_2^3+wc_2^2+w^2c_3,\\
&wc_2^3+w^2c_2+wc_3.
\end{aligned}$$

We will prove this in a number of stages. In Section 14.1 we assemble the facts that we need about the representations of Σ_4, and in Section 14.2 we deduce that the map $\kappa\colon\Omega(\Sigma_4)\longrightarrow\Omega_{\mathrm{Ch}}(\Sigma_4)$ is a bijection. We then recall some formulae for the relevant formal group law, and in Section 14.4 we use them to analyse the structure of an auxiliary scheme denoted $\mathrm{SDiv}_3^+(\mathbb{G}_0)^C$. This allows us to complete our determination of $C(K,\Sigma_4)$ in Section 14.5, with the help of some theory of Gröbner bases. We find in particular that $C(K,\Sigma_4)$ is a Gorenstein ring, which enables us to use the inner products defined in [10] to show that the map $\theta\colon C(K,\Sigma_4)\longrightarrow K^0B\Sigma_4$ is injective; this is explained in Section 14.6. We know from [5] that $K(n)^*B\Sigma_4$ is concentrated in even degrees, and it follows that $E^0B\Sigma_4$ is a free module over E^0 of rank $|\Omega(\Sigma_4)|=17$; see [11] for more details. In Section 14.7 we combine these various facts to prove the theorem.

14.1. **Representation theory.** Our first task is to understand the structure of $R(\Sigma_4)$. We have already defined the characters ϵ and ρ. It is a standard calculation that there is another irreducible character σ of dimension 2 such that the character table is as follows:

class	size	1	ϵ	σ	ρ	$\epsilon\rho$
1^4	1	1	1	2	3	3
1^22	6	1	-1	0	1	-1
2^2	3	1	1	2	-1	-1
13	8	1	1	-1	0	0
4	6	1	-1	0	-1	1

The ring structure, Adams operations and Λ-operations are described in the following table.

$$\begin{array}{lll}
\epsilon^2=1 & \psi^k(\epsilon)=\epsilon^k & \lambda^2(\sigma)=\epsilon\\
\epsilon\sigma=\sigma & \psi^2(\sigma)=1-\epsilon+\sigma & \lambda^2(\rho)=\epsilon\rho\\
\sigma^2=1+\epsilon+\sigma & \psi^3(\sigma)=1+\epsilon & \lambda^3(\rho)=\epsilon\\
\sigma\rho=\rho+\epsilon\rho & \psi^2(\rho)=1+\sigma+\rho-\epsilon\rho & \\
\rho^2=1+\sigma+\rho+\epsilon\rho & \psi^3(\rho)=1+\epsilon-\sigma+\rho. &
\end{array}$$

(The first two columns are easily checked by looking at the characters, and the last column follows using the standard formulae relating Adams operations to Λ-operations.)

Let P be a Sylow 2-subgroup (a dihedral group of order 8) and I be the kernel of the restriction map $R(\Sigma_4)\longrightarrow R(P)$; one checks that $I=(\sigma-1-\epsilon)$. Put $\tau=\epsilon\rho\in R_3^+(\Sigma_4)$; one checks that $\lambda^k(\tau)=\epsilon^k\lambda^k(\rho)$ and so $\lambda^2(\tau)=\tau$ and $\lambda^3\tau=1$. We have

$$R(\Sigma_4)/I=\mathbb{Z}\{1,\epsilon,\tau,\epsilon\tau\}=\mathbb{Z}[\epsilon,\tau]/(\epsilon^2-1,\tau^2-1-(1+\epsilon)(1+\tau)).$$

The operation ψ^k acts as the identity on this ring when k is odd, and we have

$$\begin{aligned}
\psi^2(\epsilon)&=1\\
\psi^2(\tau)&=2+\epsilon+\epsilon\tau-\tau.
\end{aligned}$$

Proposition 14.2. *The set $\Omega_{Ch}(\Sigma_4)$ can be identified with the set of pairs $(d,u) \in \Theta(1) \times \mathbb{Z}[\Theta]_3^+$ such that*

$$\begin{aligned} 2d &= 0 \\ \lambda^3(u) &= [0] \\ \psi^{-1}(u) &= u \\ \psi^2(u) + u &= 2[0] + [d] + [d]u. \end{aligned}$$

Similarly, $X_{Ch}(\Sigma_4)$ can be identified with the scheme of pairs $(d,D) \in \mathbb{G}(1) \times \mathrm{Div}_3^+(\mathbb{G})$ such that

$$\begin{aligned} 2d &= 0 \\ \lambda^3(D) &= [0] \\ \psi^{-1}(D) &= D \\ \psi^2(D) + D &= 2[0] + [d] + [d]D. \end{aligned}$$

Proof. Given a positive homomorphism $f\colon R(\Sigma_4) \longrightarrow \mathbb{Z}[\Theta]$, let $d \in \Theta$ be the element such that $f(\epsilon) = [d]$ and put $u = f(\tau)$. We know from Remark 11.2 that $f(I) = 0$ and it follows easily from our description of $R(\Sigma_4)/I$ that d and u have the properties listed. Conversely, given d and u as described, we can define a homomorphism $f\colon R(\Sigma_4)/I \longrightarrow \mathbb{Z}[\Theta]$ of additive groups by

$$\begin{aligned} f(1) &= [0] \\ f(\epsilon) &= [d] \\ f(\tau) &= D \\ f(\epsilon\tau) &= [d]D. \end{aligned}$$

It is straightforward to check that this gives a homomorphism of Λ-rings, and that these constructions give the required bijection. The argument for $X_{\mathrm{Ch}}(\Sigma_4)$ is essentially the same. □

14.2. **Generalised character theory.** We next work out the generalised character theory (as recalled in Section 11) of Σ_4. The set $\Omega(\Sigma_4)$ can be described in terms of Θ^*-sets as in Example 11.6. We can thus write $\Omega(\Sigma_4)$ as the disjoint union $\Phi_0 \amalg \ldots \amalg \Phi_4$, where

- Φ_0 consists of the set $4.0^* := 0^* \amalg 0^* \amalg 0^* \amalg 0^*$.
- Φ_1 consists of the sets $2.0^* \amalg A^*$, where $A \simeq \mathbb{Z}/2$ (so $|\Phi_1| = 2^n - 1$).
- Φ_2 consists of the sets $A^* \amalg B^*$, where $A \simeq B \simeq \mathbb{Z}/2$, and A may be equal to B. We have $|\Phi_2| = \frac{1}{2}|\Phi_1|(|\Phi_1|+1) = 2^{n-1}(2^n - 1)$.
- Φ_3 consists of the sets A^* where $A \simeq (\mathbb{Z}/2)^2$. We have $|\Phi_3| = (2^n - 1)(2^{n-1} - 1)/3$ (by counting the number of linearly independent pairs in $(\mathbb{Z}/2)^n$ and dividing by $|GL_2(\mathbb{Z}/2)| = 6$).
- Φ_4 consists of the sets A^* where $A \simeq \mathbb{Z}/4$. There are $2^{2n} - 2^n$ points in Θ of order exactly 4, and each subgroup in Φ_4 contains precisely two of these, so $|\Phi_4| = (2^{2n} - 2^n)/2 = 2^{n-1}(2^n - 1)$.

Proposition 14.3. *The map $\kappa\colon \Omega(\Sigma_4) \longrightarrow \Omega_{Ch}(\Sigma_4)$ is a bijection.*

Proof. Define

$$\Psi_i = \kappa(\Phi_i) \subseteq \Omega_{\mathrm{Ch}}(\Sigma_4).$$

Recall from Example 11.6 that $\kappa(\coprod_i m_i.A_i^*)(\pi) = \sum_i m_i[A_i]$. An easy case-by-case check shows that the sets Ψ_i are disjoint and that the maps $\kappa\colon \Phi_i \longrightarrow \Psi_i$ are bijections. It will thus be enough to show that the union of the sets Ψ_i is the whole of $\Omega_{\mathrm{Ch}}(\Sigma_4)$.

Suppose we have an element $f \in \Omega_{\mathrm{Ch}}(\Sigma_4)$, with $f(\epsilon) = [d]$ and $f(\tau) = u = [a] + [b] + [c]$ say. Let A, B and C be the cyclic subgroups generated by a, b and c respectively. Put $v = f(\pi) = [0] + [d]u = [0] + [a+d] + [b+d] + [c+d]$, and recall that this determines f, because π generates $R(\Sigma_4)$ as a Λ-ring. As

$\psi^4\tau = 3[0]$ we have $4a = 4b = 4c = 0$. By Proposition 14.2, we have

$$\begin{aligned} 2d &= 0 \\ a+b+c &= 0 \\ [a]+[b]+[c] &= [-a]+[-b]+[-c] \\ [2a]+[2b]+[2c]+[a]+[b]+[c] &= 2[0]+[d]+[a+d]+[b+d]+[c+d]. \end{aligned}$$

Suppose that $\psi^2(u) \neq 3[0]$. Without loss of generality we may assume that $2a \neq 0$ so $-a \neq a$. The third equation implies that $-a \in \{b,c\}$, so we may assume that $-a = b$. As $a+b+c = 0$ we must have $c = 0$. Recall also that $4a = 0$ so $2a = -2a$. Putting all this in the last equation and cancelling $2[0]$ gives

$$2[2a]+[a]+[-a] = 2[d]+[d+a]+[d-a].$$

Note that $2a$ and d have order 2, but a, $-a$, $d+a$ and $d-a$ do not. It follows that we must have $2a = d$ and thus $v = [0]+[a]+[2a]+[3a]$. We conclude that $f = \kappa(A^*) \in \Psi_4$.

We may thus assume that $\psi^2(u) = 3[0]$, so $2a = 2b = 2c = 0$. Suppose that $d = 0$. As $a+b+c = 0$ we see that $D := \{0,a,b,c\}$ is a subgroup of Θ, of order 2^e say (so $e \in \{0,1,2\}$). This implies that $v = 2^{2-e}[A] = \kappa(2^{2-e}.D^*)(\pi)$, so $f = \kappa(2^{2-e}.D^*) \in \Psi_0 \amalg \Psi_2 \amalg \Psi_3$.

We may thus assume that $2a = 2b = 2c = 2d = 0$ and $d \neq 0$. The equation $\psi^2(u) + u = 2[0]+[d]+[d]u$ then reduces to

$$[0]+[a]+[b]+[c] = [d]+[a+d]+[b+d]+[c+d].$$

It follows that $d \in \{a,b,c\}$ and without loss we may assume that $d = a$. Note that $c = a+b = d+b$ (because $a+b+c = 0$). If $b = 0$ this gives $c = a = d$ so $v = 3[0]+[a]$, so $f = \kappa(2.0^* \amalg A^*)$. The same argument works if $c = 0$, so we reduce to the case where $a = d \neq 0$ and b and c are also nonzero. We then have

$$v = [0]+[a+d]+[b+d]+[c+d] = 2[0]+[c]+[b] = [B]+[C],$$

so $f = \kappa(B^* \amalg C^*) \in \Phi_2$. □

14.3. **The formal group law.** Let $\mathbb{G}$ be the formal group associated to E, and let $\mathbb{G}_0$ be its restriction to the special fibre $X_0 \subset X$, or equivalently the formal group associated to K. This has a standard coordinate giving rise to a formal group law F over $\mathcal{O}_{X_0} = K^0 = \mathbb{F}_4$, which is in fact defined over $\mathbb{F}_2$. We will need the following formulae:

$$\begin{aligned} [2](x) &= x^4 \\ [-1](x) &= x + x^4 + x^{10} + x^{16} + x^{22} \pmod{x^{32}} \\ x +_F y &= x + y + x^2y^2 \pmod{x^4y^4}. \end{aligned}$$

The first of these is well-known and the second can be proved by straightforward computation; for the third, one can adapt the method of [9, Section 15] to the case $p = 2$.

14.4. **The scheme $\mathrm{SDiv}_3^+(\mathbb{G}_0)^C$.** Let C be the group (of order 2) generated by ψ^{-1}, so

$$\mathrm{SDiv}_3^+(\mathbb{G}_0)^C = \{D \in \mathrm{Div}_3^+(\mathbb{G}_0) \mid \lambda^3 D = [0] \text{ and } \psi^{-1}D = D\}.$$

We have seen that $\mathcal{O}_{\mathrm{SDiv}_3^+(\mathbb{G}_0)} = \mathbb{F}_4[\![c_2,c_3]\!]$; our next task is to determine the quotient ring $\mathcal{O}_{\mathrm{SDiv}_3^+(\mathbb{G}_0)^C}$.

Proposition 14.4. *We have*

$$\mathcal{O}_{\mathrm{SDiv}_3^+(\mathbb{G}_0)^C} = \mathbb{F}_4[\![c_2,c_3]\!]/(c_2c_3, c_3^2) = \mathbb{F}_4[\![c_2]\!] \oplus \mathbb{F}_4.c_3.$$

Proof. Put $A = \mathbb{F}_4[\![x,y,z]\!]$ and

$$\begin{aligned} d &= x +_F y +_F z \\ c_1 &= x + y + z \\ c_2 &= xy + yz + zx \\ c_3 &= xyz. \end{aligned}$$

Put $A' = A^{\Sigma_3} = \mathbb{F}_4[\![d,c_2,c_3]\!]$, so A is free of rank 6 over A'. Put $B' = A'/dA'$ and

$$B = A/dA = B' \otimes_{A'} A = \mathbb{F}_4[\![c_2,c_3]\!] = \mathcal{O}_{\mathrm{SDiv}_3^+(\mathbb{G}_0)}.$$

For any element $u \in A$ we write $\overline{u} = (\psi^{-1})^*(u)$, so $u \mapsto \overline{u}$ is a ring map and $\overline{u} = [-1](u)$ for $u \in \{x, y, z, d\}$. Put $C = B/(\overline{c}_2 - c_2, \overline{c}_3 - c_3)B$ and $C' = B'/(\overline{c}_2 - c_2, \overline{c}_3 - c_3)B' = \mathcal{O}_{\mathrm{SDiv}_3^+(\mathbb{G}_0)^C}$. The claim is that the ideal in C' generated by c_3 is free of rank one over $\mathbb{F}_4$, and that $C'/c_3C' = \mathbb{F}_4[\![c_2]\!]$, so that $C' = \mathbb{F}_4[\![c_2]\!] \oplus \mathbb{F}_4.c_3$.

We will think of $\mathrm{Div}_2^+(\mathbb{G}_0)$ as being embedded in $\mathrm{Div}_3^+(\mathbb{G}_0)$ by the map $D \mapsto D + [0]$, so

$$\mathcal{O}_{\mathrm{Div}_2^+(\mathbb{G}_0)} = \mathcal{O}_{\mathrm{Div}_3^+(\mathbb{G}_0)}/c_3 = \mathbb{F}_4[\![d, c_2]\!].$$

There is a faithfully flat map $\mathbb{G}_0 \to \mathrm{SDiv}_2^+(\mathbb{G}_0)$ sending a to $[a] + [-a]$, and clearly $\psi^{-1}([a] + [-a]) = [a] + [-a]$ so $\mathrm{SDiv}_2^+(\mathbb{G}_0) \subseteq \mathrm{SDiv}_3^+(\mathbb{G}_0)^C$. It follows that $\mathrm{Div}_2(\mathbb{G}_0) \cap \mathrm{SDiv}_3^+(\mathbb{G}_0)^C = \mathrm{SDiv}_2^+(\mathbb{G}_0)$, and thus that $C'/c_3C' = \mathbb{F}_4[\![c_2]\!]$ as claimed.

This implies that we must have $\overline{c}_2 - c_2 = c_3 r_2$ and $\overline{c}_3 - c_3 = c_3 r_3$ for some $r_2, r_3 \in B'$.

Now work in $B/(x, y, z)^7$. We have

$$\begin{aligned}
z &= \overline{x} +_F \overline{y} = x + y + x^4 + x^2y^2 + y^4 \\
\overline{x} &= x + x^4 \\
\overline{y} &= y + y^4 \\
\overline{z} &= x + y + x^2y^2 \\
c_2 &= x^2 + xy + y^2 + x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 \\
c_3 &= x^2y + xy^2 + x^5y + x^3y^3 + xy^5 \\
\overline{c}_2 - c_2 &= c_2c_3 = x^4y + xy^4 \\
\overline{c}_3 - c_3 &= c_3^2 = x^4y^2 + x^2y^4.
\end{aligned}$$

We also find that the ideal $c_3.(c_2, c_3)^2$ maps to zero in this ring. Using this, we find that $r_2 = c_2 \pmod{(c_2, c_3)^2}$ and $r_3 = c_3 \pmod{(c_2, c_3)^2}$, so $B' = \mathbb{F}_4[\![r_2, r_3]\!]$ and $B'/(r_2, r_3) = \mathbb{F}_4$. It follows that $\mathcal{O}_{\mathrm{SDiv}_3^+(\mathbb{G}_0)^C} = B'/(r_2c_3, r_3c_3) = B'/(c_2c_3, c_3^2)3$ as claimed. ∎

Now put $Y = \{D \in \mathrm{SDiv}_3^+(\mathbb{G}_0)^C \mid \psi^4 D = 3[0]\}$, and let $U \subset \mathbb{G}_0$ be the divisor $32[0]$. We know that $c_k(\psi^4 D) = c_k(D)^{16}$ so

$$\mathcal{O}_Y = \mathcal{O}_{\mathrm{SDiv}_3^+(\mathbb{G}_0)^C}/(c_1^{16}, c_2^{16}, c_3^{16}) = \mathbb{F}_4[c_2, c_3]/(c_2^{16}, c_3^2, c_2c_3).$$

We can also study Y using the maps

$$\alpha\colon U \to Y \qquad\qquad \alpha(a) = [a] + [-a] + [0]$$

$$\beta\colon \mathbb{G}_0(1)^2 \to Y \qquad\qquad \beta(a, b) = [a] + [b] + [a + b].$$

The map α gives a ring map $\alpha^*\colon \mathcal{O}_Y \to \mathbb{F}[x]/x^{32}$, with

$$\begin{aligned}
\alpha^*(c_1) &= x + \overline{x} = x^4 + x^{10} + x^{16} + x^{22} \\
\alpha^*(c_2) &= x\overline{x} = x^2 + x^5 + x^{11} + x^{17} + x^{23} \\
\alpha^*(c_3) &= 0.
\end{aligned}$$

The map β gives a ring map $\beta^*\colon \mathcal{O}_Y \to \mathbb{F}[x, y]/(x^4, y^4)$. If we put $z = x +_F y = x + y + x^2y^2$ then

$$\begin{aligned}
\beta^*(c_1) &= x + y + z = x^2y^2 \\
\beta^*(c_2) &= xy + yz + zx = x^2 + xy + y^2 + x^2y^2(x + y) \\
\beta^*(c_3) &= xyz = xy(x + y) + x^3y^3.
\end{aligned}$$

Proposition 14.5. *The maps α^* and β^* are jointly injective (in other words,* $\ker(\alpha^*) \cap \ker(\beta^*) = 0$*). Moreover, we have $c_1 = c_2^2 + c_2^8$ in $\mathcal{O}_Y$.*

Proof. Recall that $\mathcal{O}_Y = \mathbb{F}_4[c_2,c_3]/(c_2^{16},c_3^2,c_2c_3)$, so $\{c_2^i \mid 0 \le i < 16\} \amalg \{c_3\}$ is a basis for $\mathcal{O}_Y$ over $\mathbb{F}_4$. As $\alpha^*(c_2) = x^2 \pmod{x^3}$, it is easy to see that $\alpha^*(c_2^i) = x^{2i} \pmod{x^{2i+1}}$ and that these elements are linearly independent in $\mathcal{O}_U = \mathbb{F}_4[x]/x^{32}$. Moreover, we have

$$\begin{aligned}
\beta^*(1) &= 1 \\
\beta^*(c_2) &= x^2+xy+y^2+x^2y^2(x+y) \\
\beta^*(c_2^2) &= x^2y^2 \\
\beta^*(c_2^3) &= x^3y^3 \\
\beta^*(c_2^i) &= 0 && \text{for } i>3.
\end{aligned}$$

It is easy to check that $\beta^*(c_3)$ does not lie in the span of these elements, and to deduce that α^* and β^* are jointly injective as claimed. Thus, to show that $c_1 = c_2^2+c_2^8$ we need only check that $\alpha^*(c_1) = \alpha^*(c_2^2+c_2^8)$ and $\beta^*(c_1) = \beta^*(c_2^2+c_2^8)$, which is a straightforward computation. □

14.5. **The ring $C(K,\Sigma_4)$.** Consider a pair $(d,D) \in \mathbb{G}_0(1)\times Y$. This gives us a divisor $[d]D \in \mathrm{Div}_3^+(\mathbb{G}_0)$ defined over the ring

$$\mathcal{O}_{\mathbb{G}_0(1)} \otimes \mathcal{O}_Y = \mathbb{F}_4[w,c_2,c_3]/(w^4,c_2^{16},c_3^2,c_2c_3).$$

Here of course c_2 and c_3 are the usual invariants of the divisor D, but the divisor $[d]D$ also has invariants $c_k([d]D)$ lying in the above ring. In order to apply the description of $X_{\mathrm{Ch}}(\Sigma_4)$ in Proposition 14.2, we will need to understand these invariants.

Proposition 14.6. *We have*

$$\begin{aligned}
c_1([d]D) &= c_2^2+c_2^8+w+c_2^4w^2 \\
c_2([d]D) &= c_2+(1+c_2^3+c_2^9+c_3)w^2 \\
c_3([d]D) &= c_3+c_2w+(c_2^2+c_2^8)w^2+(1+c_3+c_2^3+c_2^9)w^3.
\end{aligned}$$

Proof. First recall that $u+_F v = u+v+u^2v^2 \pmod{u^4v^4}$ and $w^4=0$ so $w+_F v = w+v+w^2v^2$ for any v.

Next note that $[d]\alpha(a) = [d]([a]+[-a]+[0]) = [d+a]+[d-a]+[d]$, so $\alpha^*(c_k([d]D)) = c_k([d+a]+[d-a]+[d])$ is the k'th elementary symmetric function of $\{w+_F x, w+_F \overline{x}, w\}$, for example

$$\begin{aligned}
\alpha^*c_1([d]D) &= w+(w+x+w^2x^2)+(w+\overline{x}+w^2\overline{x}^2) \\
&= x^4+x^{10}+x^{16}+x^{22}+w+x^8w^2+x^{20}w^2 \\
&= \alpha^*(c_2^2+c_2^8)+w+\alpha^*(c_2^4)w^2.
\end{aligned}$$

By similar computations, our other two equations also become true when we apply α^*.

In the same way, we have $[d]\beta(a,b) = [d+a]+[d+b]+[d+a+b]$, so $\beta^*c_k([d]D)$ is the k'th elementary symmetric function of the list $\{w+_F x, w+_F y, w+_F x+_F y\}$, or equivalently the list

$$\{w+x+w^2x^2, w+y+w^2y^2, w+x+y+w^2x^2+w^2y^2+x^2y^2\}.$$

We thus have

$$\begin{aligned}
\beta^*c_3([d]D) &= (w+x+w^2x^2)(w+y+w^2y^2)(w+x+y+w^2x^2+w^2y^2+x^2y^2) \\
&= (x^2y+xy^2+x^3y^3)+(x^2+xy+y^2+x^3y^2+x^2y^3)w+ \\
&\quad\ x^2y^2w^2+(1+x^2y+xy^2)w^3 \\
&= \beta^*(c_3)+\beta^*(c_2)w+\beta^*(c_2^2+c_2^8)w^2+\beta^*(1+c_3+c_2^3+c_2^9)w^3.
\end{aligned}$$

By similar computations, our other two equations also become true when we apply β^*. As α^* and β^* are jointly injective, it follows that our equations hold in $\mathcal{O}_{\mathbb{G}_0(1)\times Y}$ as claimed. □

Proposition 14.7. *Let J be the ideal in $\mathbb{F}_4[w, c_2, c_3]$ generated by the elements*

$$\begin{aligned} &w^4\ ,\ c_3^2\ ,\ c_2c_3\ , \\ &c_2^4 + w^2c_2^3 + wc_2^2 + w^2c_3, \\ &wc_2^3 + w^2c_2 + wc_3. \end{aligned}$$

Then $C(K, \Sigma_4) = \mathbb{F}_4[w, c_2, c_3]/J$. Moreover, the following monomials form a basis for this ring over $\mathbb{F}_4$, so it has dimension 17.

1	c_2	c_2^2	c_2^3	c_3
w	wc_2	wc_2^2		wc_3
w^2	w^2c_2	$w^2c_2^2$		w^2c_3
w^3	w^3c_2	$w^3c_2^2$		w^3c_3

Proof. Proposition 14.2 is equivalent to the statement that

$$X(\Sigma_4) = \{(d, D) \in \mathbb{G}_0(1) \times Y \mid D + \psi^2(D) = 2[0] + [d] + [d]D\}.$$

This means that $C(K, \Sigma_4)$ is the largest quotient of $\mathcal{O}_{\mathbb{G}_0(1)\times Y}$ over which we have $g(t) = 0$, where

$$g(t) = f_D(t)f_{\psi^2D}(t) - t^2(t + w)f_{[d]D}(t).$$

Here we write $f_D(t) = t^3 + c_1(D)t^2 + c_2(D)t + c_3(D)$ and similarly for our other divisors. As usual we write c_k for $c_k(D)$, and we recall from Proposition 14.5 that $c_1 = c_2^2 + c_2^8$. We also recall that $c_k(\psi^2D) = c_k(D)^4$, so that

$$f_{\psi^2D}(t) = t^3 + c_1^4t^2 + c_2^4t + c_3^4 = t^3 + c_2^8t^2 + c_2^4t.$$

The polynomial $f_{[d]D}(t) = \sum_{k=0}^3 c_k([d]D)t^{3-k}$ can be read off from Proposition 14.6. Putting all this together and expanding it out, we find that $g(t) = \sum_{k=1}^4 r_kt^{6-k}$, where

$$\begin{aligned} r_1 &= c_2^8 + c_2^4w^2 \\ r_2 &= c_2^4w^3 + (c_2^9 + c_2^3 + c_3)w^2 + (c_2^8 + c_2^2)w + (c_2^{10} + c_2^4) \\ r_3 &= (c_2^8 + c_2^2)w^2 + (c_2^{12} + c_2^9 + c_2^6) \\ r_4 &= (c_2^8 + c_2^2)w^3 + c_2w^2 + c_3w + c_2^5. \end{aligned}$$

We thus have

$$C(K, \Sigma_4) = \mathbb{F}_4[w, c_2, c_3]/(w^4, c_2^{16}, c_3^2, c_2c_3, r_1, r_2, r_3, r_4).$$

As $1 + c_2^6 + c_2^{12} + w^3$ is invertible, we can replace r_2 by

$$r_2' := (1 + c_2^6 + c_2^{12} + w^3)r_2 = c_2^4 + w^2c_2^3 + wc_2^2 + w^2c_3,$$

which is one of the relations in the statement of the theorem. As $w^4 = 0$ we have $(r_2')^2 = r_1$ and $(r_2')^4 = c_2^{16}$, so the relations r_1 and c_2^{16} are redundant. Similarly, we can replace r_4 by the relation

$$r_4' := r_4 + (c_2 + w^2 + c2^4w^3)r_2' = wc_2^3 + w^2c_2 + wc_3,$$

which is another of the relations in the statement of the theorem. One can check that

$$r_3 = (1 + c_2^3 + c_2^6)(c_2^2(1 + (1 + c_2^6)(w^3 + c_2^2w^2))r_2' + c_2r_4'),$$

so r_3 is redundant. We deduce that

$$C(K, \Sigma_4) = \mathbb{F}_4[w, c_2, c_3]/(w^4, c_3^2, c_2c_3, r_4', r_2')$$

as claimed.

We next show that the 17 monomials listed form a basis for this quotient ring. We order the set of monomials in w, c_2 and c_3 by saying that $c_2^ic_3^jw^k < c_2^{i'}c_3^{j'}w^{k'}$ iff $i < i'$ or ($i = i'$ and $j < j'$) or ($i = i'$ and $j = j'$ and $k < k'$). We claim that our relations form a Gröbner basis for J with respect to this ordering. We first recall briefly what this means. The list of leading terms of our relations is $(w^4, c_3^2, c_2c_3, c_2^3w, c_2^4)$. A polynomial is said to be *top-reducible* if any of its monomials is divisible by one of these leading terms; if so, we can subtract off a multiple of the corresponding relation to cancel the monomial, a process called *top-reduction*. Clearly, if a polynomial can be reduced to zero by iterated top-reduction then it must lie

in J, but the converse need not hold for an arbitrary list of generators of an arbitrary ideal. Let a and b be any two of our relations, let a' and b' be their leading terms, and let c' be the greatest common divisor of a' and b'. The corresponding *syzygy* is the element $c := (a'/c')b - (b'/c')a \in J$. To say that our relations form a Gröbner basis means precisely that all these syzygies can be reduced to zero by iterated top-reduction. This can be checked by direct computation. For example, the syzygy of r'_4 and r'_2 is the element $c_2r'_4 - wr'_2 = c_2^3w^3 + c_2c_3w + c_3w^3$. The first monomial is divisible by the leading term of r'_4, so we can top-reduce by subtracting $w^2r'_4$ to get $c_2c_3w + c_2w^4$. We can then do two more top-reductions by subtracting w times the relation c_2c_3 and c_2 times the relation w^4 to get 0, as required. Now observe that the 17 monomials listed in the statement of the theorem are precisely those that are not top-reducible. It follows from the theory of Gröbner bases that they form a basis for $C(E, \Sigma_4)$, as claimed. ☐

Corollary 14.8. *$C(K, \Sigma_4)$ is a Gorenstein ring, and the element w^3c_3 generates the socle.*

Proof. One sees easily from the relations listed that w, c_2 and c_3 annihilate w^3c_3, so w^3c_3 lies in the socle. Now let a be an arbitrary element of the socle. It will be convenient to put $e = c_2^3 + wc_2 + c_3$ (so that $we = c_3e = 0$) and to use the basis given in the Proposition but with c_2^3 replaced by e. Using the equation $wa = 0$ we see immediately that a lies in the span of $\{w^3, w^3c_2, w^3c_2^2, e, w^3c_3\}$. Using the equation $c_3a = 0$ and the fact that $c_3c_2 = c_3e = c_3^2 = 0$ we find that the coefficient of w^3 is zero, so $a = \alpha w^3c_2 + \beta w^3c_2^2 + \gamma e + \delta w^3c_3$ say. One can check that $w^3c_2^3 = w^3c_3$ and $c_2e = w^3c_2$, so

$$0 = c_2a = \alpha w^3c_2^2 + \beta w^3c_3 + \gamma w^3c_2,$$

so $\alpha = \beta = \gamma = 0$, so $a = \delta w^3c_3$. This shows that the socle is one-dimensional, so the ring is Gorenstein as claimed. ☐

14.6. **A transfer argument.**

Proposition 14.9. *The map $\theta \colon C(K, \Sigma_4) \to K^0B\Sigma_4$ is injective.*

Proof. Note that every nontrivial ideal in $C(K, \Sigma_4)$ contains the socle, so it will suffice to show that the socle is not contained in $\ker(\theta)$, or equivalently that $w^3c_3 \neq 0$ in $K^0B\Sigma_4$. Let P be the Sylow subgroup in Σ_4; it will be enough to show that w^3c_3 has nontrivial image in K^0BP. Put $V = P \cap A_4$; one can check that this consists of the identity and the three transposition pairs, so it is isomorphic to C_2^2. Recall that the series $\langle 2\rangle(x)$ is defined to be $[2](x)/x$, which in our case is just x^3. As w is the Euler class of ϵ and $V = \ker(\epsilon \colon P \to C_2)$, standard arguments show that $\mathrm{tr}_V^P(1) = \langle 2\rangle(w) = w^3$. This means that $w^3c_3 = \mathrm{tr}_V^P(c_3)$. To see that this is nonzero, we use the canonical bilinear form on K^0BP defined in [10]. This satisfies Frobenius reciprocity, so $(\mathrm{tr}_V^P(c_3), 1)_P = (c_3, 1)_V$. If we let x and y be the Euler classes of two of the nontrivial characters of V, then $K^0BV = \mathbb{F}_4[x, y]/(x^4, y^4)$ and the Euler class of the third character is $x +_F y = x + y + x^2y^2$. One checks that the restriction of ρ to V is the regular representation minus the trivial representation, which is the sum of the three nontrivial characters. This implies that the restriction of c_3 to V is $xy(x +_F y) = x^2y + xy^2 + x^3y^3$. Using [10, Corollary 9.3] we see that $(x^iy^j, 1)_V$ is 1 if $i = j = 3$ and 0 otherwise, so $(c_3, 1)_V = 1$. As $(w^3c_3, 1)_P = (c_3, 1)_V = 1$ we see that $w^3c_3 \neq 0$, as claimed. ☐

14.7. **The proof of Theorem 14.1.** This is now easy. We know from [11] that $E^0B\Sigma_4$ is a free module of finite rank over E^0. It follows by well-known arguments that $K^0B\Sigma_4 = (E^0B\Sigma_4)/I_2$, which is free of the same rank over $K^0 = E^0/I_2 = \mathbb{F}_4$. The rank is also the same as the rank of $L \otimes E^0B\Sigma_4$ over L, and generalised character theory tells us that this is equal to $|\Omega(\Sigma_4)| = 17$. Thus, the source and target of the map $\theta \colon C(K, \Sigma_4) \to K^0B\Sigma_4$ both have rank 17 over $\mathbb{F}_4$ and Proposition 14.9 tells us that the map is injective, so it must be an isomorphism. Now consider the map $\theta \colon C(E, \Sigma_4) \to E^0B\Sigma_4$. This is an isomorphism modulo I_2, so by Nakayama's lemma it is surjective. As $E^0B\Sigma_4$ is free it is a split surjection, so $C(E, \Sigma_4) = E^0B\Sigma_4 \oplus N$ say. This implies that $C(K, \Sigma_4) = K^0B\Sigma_4 \oplus N/I_2N$, so by counting ranks we see that $N/I_2N = 0$, so by Nakayama again we see that $N = 0$. Thus $C(E, \Sigma_4) = E^0B\Sigma_4$ as claimed. We know from Proposition 14.7 that our list of 17 monomials is a basis for $K^0B\Sigma_4$ over $\mathbb{F}_4$, and it now follows that it is also a basis for $E^0B\Sigma_4$ over E^0.

15. Extraspecial p-groups

In this section we define a class of "extraspecial" p-groups (where p is an odd prime), and show that for these groups the map $\kappa\colon \Omega(G) \longrightarrow \Omega_{\mathrm{Ch}}(G)$ is injective but not surjective. It follows using Theorem 11.4 that the map $\theta\colon C(E,G) \longrightarrow E^0BG$ cannot be an isomorphism. We have not investigated the situation more deeply than this.

Let V be an elementary Abelian p-group of rank $2d$ equipped with a nondegenerate alternating form $b\colon V \times V \longrightarrow \mathbb{F}_p$. We will say that a subspace $W \leq V$ is *isotropic* if $b(u,v) = 0$ for all $u, v \in W$.

Let G be the set $\mathbb{F}_p \times V$ with the group operation $(x,u).(y,v) = (x + y + b(u,v), u + v)$. This has order p^{2d+1} and exponent p, and it fits in a central extension

$$Z = \mathbb{F}_p \xrightarrow{j} G \xrightarrow{q} V.$$

In fact Z is the centre of G, and the non-central conjugacy classes are the fibres of q over $V \setminus \{0\}$, so they all have order p. This gives $p + p^{2d} - 1$ conjugacy classes altogether.

We can evidently view $R(V) = \mathbb{Z}[V^*]$ as a sub Λ-ring of $R(G)$.

Definition 15.1. For any nontrivial character $\zeta\colon Z \longrightarrow S^1$, let $\phi(\zeta)$ be the class function on G defined by

$$\phi(\zeta)(g) = \begin{cases} p^d\zeta(g) & \text{if } g \in Z \\ 0 & \text{otherwise.} \end{cases}$$

We also write ρ_V for the regular representation of V, and ρ_G for the regular representation of G.

The following result is standard, but we give a proof for completeness.

Proposition 15.2. *For each $\zeta \in Z^* \setminus \{1\}$, the class function $\phi(\zeta)$ is an irreducible character. Moreover, we have*

$$R(G) = \mathbb{Z}[V^*] \oplus \mathbb{Z}\{\phi(\zeta) \mid \zeta \in Z^* \setminus \{1\}\}.$$

Proof. Choose a maximal isotropic subspace $W \leq V$, so $W \simeq \mathbb{F}_p^d$. Put $H = q^{-1}W \leq G$, which is isomorphic to $Z \times W$ as a group because W is isotropic. Let $r\colon H \longrightarrow Z$ be the projection and put $\sigma = \mathrm{ind}_H^G \pi^*\zeta$. We claim that $\sigma = \phi(\zeta)$. To see this, first note that H is normal in G, so $\sigma(g) = 0$ for $g \notin H$. Next, suppose that $g \in H \setminus Z$, say $g = (x,w)$ with $w \in W \setminus \{0\}$. Let U be such that $V = W \oplus U$, so $U \simeq \mathbb{F}_p^d$ and $(0,u)^{-1}(x,w)(0,u) = (x - 2b(u,w), w)$. From the definitions we see that $\sigma(x,w) = \sum_u \zeta(x - 2b(u,w), 0)$. The map $u \mapsto (x - 2b(u,w), 0)$ is a surjection from U to Z, each of whose fibres has the same order, and $\zeta\colon Z \longrightarrow S^1$ is a nontrivial homomorphism; it follows easily that $\sigma(x,w) = 0$, as required. Finally, suppose that $g \in Z$, say $g = (x,0)$. Then $(0,u)^{-1}(x,0)(0,u) = (x,0)$ so $\sigma(x,0) = \sum_u \zeta(x,0) = p^d\zeta(x,0)$. This shows that $\sigma = \phi(\zeta)$ as claimed, so $\phi(\zeta)$ is a character. One checks easily that $\langle\phi(\zeta),\phi(\zeta)\rangle = |G|^{-1}\sum_{z\in Z} p^{2d} = 1$, so $\phi(\zeta)$ is irreducible. As ζ runs over $Z^* \setminus \{1\}$ this gives $p-1$ distinct irreducibles of degree p^d, and V^* gives a further p^{2d} distinct irreducibles of degree 1. We have seen that G has $p^{2d}+p-1$ conjugacy classes and thus $p^{2d}+p-1$ irreducible characters, so our list is complete. It follows that $R(G) = \mathbb{Z}[V^*] \oplus \mathbb{Z}\{\phi(\zeta) \mid \zeta \in Z^* \setminus \{1\}\}$ as claimed. ∎

Lemma 15.3. *Let C be cyclic of order p. Then*

$$\lambda^k(p^{d-1}\rho_C) = \begin{cases} \binom{p^{d-1}}{k/p} + \frac{1}{p}\left(\binom{p^d}{k} - \binom{p^{d-1}}{k/p}\right)\rho_C & \textit{if } p|k \\ \frac{1}{p}\binom{p^d}{k}\rho_C & \textit{otherwise} \end{cases}$$

Proof. Let χ be a generator of C^*, so $R(C) = \mathbb{Z}[\chi]/(\chi^p - 1)$ and $\rho_C = \sum_{j=0}^{p-1}\chi^j$. We have $\chi\rho_C = \rho_C$ and so $\lambda^k(\rho_C) = \lambda^k(\chi\rho_C) = \chi^k\lambda^k(\rho_C)$. If $0 < k < p$ then χ^k is also a generator, and it follows that $\lambda^k(\rho_C)$ is an integer multiple of ρ_C. On the other hand, it is easy to check that $\lambda^0(\rho_C) = \lambda^p(\rho_C) = 1$. If we put $A = \mathbb{Z}\{1,\rho_C\}$ then A is a subring of $R(C)$ (with $\rho_C^2 = p\rho_C$) and $\lambda_t(\rho_C) \in A[t]$ so $\lambda_t(p^{d-1}\rho_C) = \lambda_t(\rho_C)^{p^{d-1}}$ also lies in $A[t]$, say $\lambda^k(\rho_C) = n_k + m_k\rho_C$. Moreover, if we work mod ρ_C we have $\lambda_t(\rho_C) \cong 1 + t^p$ so $\lambda_t(p^{d-1}\rho_C) \cong (1+t^p)^{p^{d-1}}$. Thus, if p divides k then $n_k = \binom{p^{d-1}}{k/p}$, and if p does not divide k then $n_k = 0$. Moreover, by counting dimensions we see that $n_k + pm_k = \binom{p^d}{k}$ for all k. The lemma now follows easily. ∎

Proposition 15.4.

$$
\begin{aligned}
\chi\phi(\zeta) &= \phi(\zeta) \\
\phi(\zeta)\phi(\xi) &= \begin{cases} \rho_V & \textit{if } \zeta\xi = 1 \\ p^d\phi(\zeta\xi) & \textit{otherwise} \end{cases} \\
\psi^k\chi &= \chi^k \\
\psi^k\phi(\zeta) &= \begin{cases} p^d & \textit{if } p|k \\ \phi(\zeta^k) & \textit{otherwise} \end{cases} \\
\lambda^k(\phi(\zeta)) &= \begin{cases} \binom{p^{d-1}}{k/p} + \frac{1}{p^{2d}}\left(\binom{p^d}{k} - \binom{p^{d-1}}{k/p}\right)\rho_V & \textit{if } p|k \\ \frac{1}{p^d}\binom{p^d}{k}\phi(\zeta^k) & \textit{otherwise} \end{cases}
\end{aligned}
$$

Proof. Everything except for $\lambda^k(\phi(\zeta))$ can be done by easy manipulation of characters. For the remaining case, it suffices to check that the claimed equations hold when restricted to any cyclic subgroup $C \leq G$. First consider the case $C = Z$, so ρ_V restricts on C to the trivial representation of degree p^{2d}. Then $\phi(\zeta)$ becomes $p^d\zeta$, so $\lambda^k\phi(\zeta)$ becomes $\binom{p^d}{k}\zeta^k$. Using this, it is easy to check that the equations hold when restricted to Z.

Now suppose instead that $C \leq G$ is a cyclic group not contained in Z (which implies that $|C| = p$). Then $\rho_V|_C = p^{2d-1}\rho_C$ and $\phi(\xi)|_C = p^{d-1}\rho_C$ for all $\xi \in Z^* \setminus \{1\}$. Using Lemma 15.3 we deduce that our equations for $\lambda^k(\phi(\zeta))$ are correct when restricted to C, as required. ∎

Definition 15.5. For any homomorphism $\alpha\colon V^* \to \Theta$, put

$$c_\alpha = \sum_{\chi \in V^*} [\alpha(\chi)] \in \mathbb{Z}[\Theta(1)]^+_{p^{2d}}$$

and

$$U_\alpha = \{u \in \mathbb{Z}[\Theta(1)]^+_{p^d} \mid u\psi^{p-1}(u) = c_\alpha\}.$$

We also put $U = \{(\alpha, u) \mid u \in U_\alpha\}$.

Theorem 15.6. *There is a natural bijection $\Omega_{Ch}(G) = U$. The map $\kappa\colon \Omega(G) \to U$ is injective, and the image is the set of pairs $(\alpha, u) \in U$ such that the image of the dual map $\alpha^*\colon \Theta^* \to V$ is isotropic.*

The proof will follow after a lemma.

Lemma 15.7. *Let $\alpha\colon V^* \to \Theta$ be a homomorphism with image A of order p^e. If $e > d$ then $U_\alpha = \emptyset$. If $e \leq d$ then U_α is the set of elements of the form $u = p^{d-e}\sum_{c\in C}[c]$, where C runs over the cosets of A in $\Theta(1)$.*

Proof. Put $c'_\alpha = \sum_{a\in A}[a] \in \mathbb{Z}[\Theta(1)]^+_{p^e}$ so that $c_\alpha = p^{2d-e}c'_\alpha$. Suppose $u \in U_\alpha$ and that $b \in u$. Put $v = [-b]u$, so $v \in U_\alpha$ and $0 \in v$. Thus $0 \in \psi^{p-1}(v)$ also, so $v \leq v\psi^{p-1}(v) = c_\alpha = p^{2d-e}c'_\alpha$, so we can write $v = \sum_{a\in A} n_a[a]$ for suitable natural numbers n_a. By looking at the multiplicity of $[0]$ in the equation $v\psi^{p-1}(v) = p^{2d-e}c'_\alpha$ we see that $\sum_a n_a^2 = p^{2d-e}$. On the other hand, as $v \in \mathbb{Z}[\Theta(1)]^+_{p^d}$ we have $\sum_a n_a = p^d$. It follows that

$$\sum_a (n_a - p^{d-e})^2 = \sum_a n_a^2 - 2p^{d-e}\sum_a n_a + p^{2d-2e}\sum_a 1 = p^{2d-e} - 2p^{2d-e} + p^{2d-e} = 0,$$

so $n_a = p^{d-e}$ for all a. If we now let C be the coset $b + A$ we find that $u = p^{d-e}\sum_{c\in C}[c]$. Conversely, it is trivial to check that any element of this form lies in U_α. ∎

Proof of Theorem 15.6. Let ζ be the usual character $x \mapsto e^{2\pi i x/p}$ of $Z = \mathbb{Z}/p$. Given $f\colon R(G) \to \mathbb{Z}[\Theta]$ it is clear that the restriction of f to $R(V) = \mathbb{Z}[V^*]$ gives a homomorphism $\alpha\colon V^* \to \Theta(1) \leq \Theta$, and we put $u = f(\phi(\zeta)) \in \mathbb{Z}[\Theta]^+_{p^d}$. As f is a Λ-ring homomorphism we have

$$u\psi^{p-1}(u) = f(\phi(\zeta)\phi(\zeta^{p-1})) = f(\rho_V) = \sum_\chi [\alpha(\chi)],$$

so $(\alpha, u) \in U$.

Conversely, suppose we start with $(\alpha, u) \in U$. Let e and C be as in Lemma 15.7. We define a homomorphism $f\colon R(G) \to \mathbb{Z}[\Theta]$ of additive groups by $f(\chi) = [\alpha(\chi)]$ for $\chi \in V^*$ and

$$f(\phi(\zeta^k)) = \psi^k(u) = p^{d-e} \sum_{c \in kC} [c]$$

for $k \in \mathbb{Z} \setminus p\mathbb{Z}$. It is easy to check that this is a ring homomorphism that sends $R^+_k(G)$ to $\mathbb{Z}[\Theta]^+_k$ and commutes with the Adams operations. As $\mathbb{Z}[\Theta]$ is torsion free it follows that f commutes with Λ-operations as well, so $f \in \Omega_{\mathrm{Ch}}(G)$. Clearly these constructions give the required bijection $\Omega_{\mathrm{Ch}}(G) = U$.

Now suppose we have a homomorphism $\mu\colon \Theta^* \to G$. Then $\mu(u) = (\omega(u), \sigma(u))$ for some functions $\omega\colon \Theta^* \to \mathbb{F}_p$ and $\sigma\colon \Theta^* \to V$. As μ and the projection $q\colon G \to V$ are homomorphisms we see tht σ is a homomorphism. Let $W \leq V$ be the image of σ, and put $e = \dim_{\mathbb{F}_p} W$. As the image of μ must be commutative, it is not hard to see that W is isotropic, so $e \leq d$. As $q^{-1}W \simeq \mathbb{F}_p \times W$ as groups, we see that ω is also a homomorphism. If we conjugate (ω, σ) by $(x, u) \in G$ we get the homomorphism $(\omega + \tau, \sigma)$ where $\tau(t) = 2b(u, \sigma(t))$. As b is a perfect pairing, τ can be any map $\Theta^* \to \mathbb{F}_p$ that factors through σ, so (ω, σ) is conjugate to (ω', σ) if and only if $\omega|_{\ker(\sigma)} = \omega'|_{\ker(\sigma)}$. Now let $\sigma^*\colon V^* \to \Theta$ be the dual of σ and put $A = \sigma^*(V^*)$, so $|A| = p^e$. We also have a map $\omega^*\colon \mathbb{F}_p^* \to \Theta$ and thus a point $t = \omega^*(\zeta) \in \Theta(1)$. In $R(\mathbb{F}_p \times W) = \mathbb{Z}[\mathbb{F}_p^*] \otimes \mathbb{Z}[W^*]$ we have

$$\phi(\zeta)|_{\mathbb{F}_p \times W} = p^{d-e}\zeta \otimes \rho_W = p^{d-e} \sum_{\xi \in W^*} \zeta \otimes \xi,$$

and it follows that $\mu^*\phi(\zeta) = p^{d-e}\sum_{a \in A}[t + a] \in \mathbb{Z}[\Theta]$. Thus, if we write $[\mu]$ for the conjugacy class of μ then $\kappa[\mu] = (\sigma^*, p^{d-e}\sum_{a \in A}[t + a]) \in U$. It follows that $\kappa[\mu]$ determines σ, and it also determines t modulo A, so it determines ω modulo $\sigma^*(\mathrm{Hom}(V, \mathbb{F}_p))$, so it determines the conjugacy class $[\mu]$. This proves that κ is injective as claimed. We leave it to the reader to check that the image is as described. □

16. An apparently more precise approach

There are some senses in which the Λ-operations do not capture all possible information about the representation theory of G, and it is reasonable to wonder whether a more accurate approximation to $X(G)$ could be defined by taking more information into account. In this section we show that this is not the case: we construct an approximation $Y(G)$ using all possible operations, and show that it is the same as $X_{\mathrm{Ch}}(G)$.

Definition 16.1. Let $\mathcal{G}$ be the category of Lie groups and continuous homomorphisms, and let $\overline{\mathcal{G}}$ be the quotient category in which conjugate homomorphisms are identified. Let $\mathcal{N}$ be the set of finite sequences $\underline{n} = (n_1, \ldots, n_r)$ with $n_i \in \mathbb{N}$. For $\underline{n} \in \mathcal{N}$ we put $\mathrm{GL}(\underline{n}) = \prod_i \mathrm{GL}(n_i, \mathbb{C})$ and

$$R(\underline{n}, G) = \prod_i R^+_{n_i}(G) = \overline{\mathcal{G}}(G, \mathrm{GL}(\underline{n})).$$

We make $\mathcal{N}$ into a category by putting $\mathcal{N}(\underline{n}, \underline{m}) = \mathcal{G}(\mathrm{GL}(\underline{n}), \mathrm{GL}(\underline{m}))$, and we let $\overline{\mathcal{N}}$ be the category with the same objects and with morphisms $\overline{\mathcal{N}}(\underline{n}, \underline{m}) = \overline{\mathcal{G}}(\mathrm{GL}(\underline{n}), \mathrm{GL}(\underline{m}))$; clearly this gives a covariant functor $\underline{n} \mapsto R(\underline{n}, G)$ from $\overline{\mathcal{N}}$ to sets.

Next, let $T(\underline{n})$ be the evident maximal compact torus in $\mathrm{GL}(\underline{n})$, so $T(\underline{n}) \simeq \prod_{j=1}^N S^1$ where $N = \sum_{i=1}^r n_i$. Let $W(\underline{n})$ be the Weyl group of $T(\underline{n})$, so $W(\underline{n}) \simeq \prod_i \Sigma_{n_i}$. We can thus form the scheme

$$D(\underline{n}) = \mathrm{Hom}(T(\underline{n})^*, \mathbb{G})/W(\underline{n}) = \prod_i \mathrm{Div}^+_{n_i}(\mathbb{G}).$$

By elementary arguments in representation theory we see that any homomorphism $f\colon \mathrm{GL}(\underline{n}) \to \mathrm{GL}(\underline{m})$ is conjugate to one that sends $T(\underline{n})$ into $T(\underline{m})$, and that the resulting map $T(\underline{n}) \to T(\underline{m})$ is unique up to the action of $W(\underline{m})$. Using this, it is easy to make the assignment $\underline{n} \mapsto D(\underline{n})$ into a functor $\overline{\mathcal{N}} \to \widehat{\mathcal{X}}_X$.

Finally, we define a functor $Y(G)$ from discrete $\mathcal{O}_X$-algebras to sets by putting

$$Y(G)(A) = [\overline{\mathcal{N}}, \mathrm{Sets}](R(-,G), D(-)(A)).$$

Theorem 16.2. *There is a natural isomorphism $Y(G) \simeq X_{Ch}(G)$.*

Before proving this, we relate $Y(G)$ to an auxiliary model involving unitary groups rather than general linear groups.

Definition 16.3. Let $\widetilde{\mathcal{G}}$ be the quotient of $\mathcal{G}$ in which homomorphisms $u, v\colon U \to V$ are identified if $u|_K$ is conjugate to $v|_K$ for every compact subgroup $K \leq U$. (For example, the homomorphism $u\colon \mathrm{GL}(1) \to \mathrm{GL}(1)$ given by $u(z) = |z|$ becomes trivial in $\widetilde{\mathcal{G}}$.) Let $\widetilde{\mathcal{N}}$ be the category with the same objects as $\mathcal{N}$ and morphisms $\widetilde{\mathcal{N}}(\underline{n}, \underline{m}) = \widetilde{\mathcal{G}}(\mathrm{GL}(\underline{n}), \mathrm{GL}(\underline{m}))$. As G and $T(\underline{n})$ are compact, it is clear that the functors $R(-,G)$ and $D(-)$ factor through $\widetilde{\mathcal{N}}$, and thus that

$$Y(G)(A) = [\widetilde{\mathcal{N}}, \mathrm{Sets}](R(-,G), D(-)(A)).$$

Lemma 16.4. *Let K be a compact Lie group, and let $v, w\colon K \to U(d)$ be continuous homomorphisms. If v and w are conjugate in $\mathrm{GL}(d)$, then they are conjugate in $U(d)$.*

Proof. The statement can easily be translated as follows: Let V and W be finite-dimensional vector spaces over $\mathbb{C}$ equipped with actions of G and invariant Hermitian inner products. Then if there exists an equivariant isomorphism $f\colon V \to W$, then f can be chosen to preserve the inner products.

To see this, we first recall some facts about invariant Hermitian products. For any complex vector space V we let $\overline{V}$ be the same set with the conjugate action of $\mathbb{C}$, and let $\overline{V}^*$ be the dual of $\overline{V}$. The set of Hermitian products β on V bijects with the set of isomorphisms $\beta'\colon V \to \overline{V}^*$ satisfying certain symmetry and positivity conditions. For any representation V one can always choose an invariant Hermitian product so V is equivariantly isomorphic to $\overline{V}^*$. For each irreducible representation S we fix a Hermitian product β_S on S; Schur's lemma implies that $\mathrm{Hom}_K(S, \overline{S}^*) = \mathbb{C}\beta'_S$ and that any other invariant Hermitian product is a positive scalar multiple of β_S.

Now let β be a Hermitian product on V and suppose that $V = V_0 \oplus V_1$ and $\mathrm{Hom}_K(V_1, V_0) = 0$. Then $\mathrm{Hom}_K(V_1, \overline{V}_0^*) = 0$ and

$$\mathrm{Hom}_K(V_0, \overline{V}_1^*) = \mathrm{Hom}_K(\overline{V}_1^*, V_0)^* = \mathrm{Hom}_K(V_1, V_0)^* = 0$$

so the equivariant isomorphism $\beta'\colon V \to \overline{V}^*$ must have the form $\beta'_0 \oplus \beta'_1$ for some $\beta'_i\colon V_i \to \overline{V}_i^*$. This implies that V_0 and V_1 are orthogonal with respect to β.

Now let $S_1, \ldots, S_t$ be the distinct irreducible representations that occur in V. Then there is a unique decomposition $V = V_1 \oplus \ldots \oplus V_t$, where $V_i \simeq \mathbb{C}^{d_i} \otimes S_i$ for some d_i and thus $\mathrm{Hom}_K(V_i, V_j) = 0$ when $i \neq j$. By the previous paragraph, the subspaces V_i are orthogonal to each other. As $\mathrm{Hom}_K(S_i, \overline{S}_i^*) = \mathbb{C}\beta_{S_i}$, we find that the restriction of β to V_i has the form $\gamma_i \otimes \beta_{S_i}$ for some Hermitian product γ_i on $\mathbb{C}^{d_i}$. By Gram-Schmidt, the space $(\mathbb{C}^{d_i}, \gamma_i)$ is isomorphic to $\mathbb{C}^{d_i}$ with its usual Hermitian product, so $(V_i, \beta|_{V_i})$ is equivariantly and isometrically isomorphic to the orthogonal direct sum of d_i copies of (S_i, β_{S_i}). This means that the numbers d_i determine the *isometric* isomorphism type of V, and the lemma follows immediately. □

Lemma 16.5. *There are natural bijections*

$$\widetilde{\mathcal{N}}(\underline{n}, \underline{m}) = \overline{\mathcal{G}}(U(\underline{n}), \mathrm{GL}(\underline{m})) = \overline{\mathcal{G}}(U(\underline{n}), U(\underline{m})),$$

where $U(\underline{n}) = \prod_i U(n_i) \leq \mathrm{GL}(\underline{n})$.

Proof. It is easy to reduce to the case where the list $\underline{m}$ has length 1, say $\underline{m} = (d)$. As any representation of $U(\underline{n})$ admits a Hermitian inner product, we see that the map $\overline{\mathcal{G}}(U(\underline{n}), U(d)) \to \overline{\mathcal{G}}(U(\underline{n}), \mathrm{GL}(d))$ is surjective. It is also injective by Lemma 16.4. Similarly, by considering invariant Hermitian products we see that if K is compact and $u\colon K \to \mathrm{GL}(\underline{n})$ then u is conjugate to a homomorphism $K \to U(\underline{n})$. By applying this to the

inclusion map, we see that any compact subgroup of $\mathrm{GL}(\underline{n})$ is conjugate to a subgroup of $U(\underline{n})$. It follows that any two homomorphisms $v, w\colon \mathrm{GL}(\underline{n}) \to \mathrm{GL}(d)$ are identified in $\widetilde{\mathcal{G}}(\mathrm{GL}(\underline{n}), \mathrm{GL}(d))$ iff their restrictions to $U(\underline{n})$ are conjugate, so we have a well-defined and injective restriction map $\widetilde{\mathcal{N}}(\underline{n}, d) \to \overline{\mathcal{G}}(U(\underline{n}), \mathrm{GL}(d))$. It is an easy consequence of the theory of roots and so on that any representation of $U(\underline{n})$ extends uniquely to a complex-analytic representation of $\mathrm{GL}(\underline{n})$, so our restriction map is also surjective. □

Proof of Theorem 16.2. Consider a point $g \in Y(G)(A)$, in other words a natural transformation $g_{\underline{n}}\colon R(\underline{n}, G) \to D(\underline{n})(A)$ for $\underline{n} \in \mathcal{N}$. By putting together the maps

$$g_d\colon R^+_d(G) = R(d, G) \to D(d)(A) = \mathrm{Div}^+_d(\mathbb{G})(A),$$

we get a function $f\colon R^+(G) \to \mathrm{Div}^+(\mathbb{G})(A)$. Next, for any $d, e \geq 0$ we have projections $\mathrm{GL}(d) \leftarrow \mathrm{GL}(d, e) \to \mathrm{GL}(e)$ and we can use the resulting maps to identify $R((d, e), G)$ with $R^+_d(G) \times R^+_e(G)$ and $D(d, e)(A)$ with $\mathrm{Div}^+_d(\mathbb{G})(A) \times \mathrm{Div}^+_e(\mathbb{G})(A)$ and $g_{(d,e)}$ with $g_d \times g_e$. There are evident maps $\oplus\colon \mathrm{GL}(d, e) \to \mathrm{GL}(d + e)$ and $\otimes\colon \mathrm{GL}(d, e) \to \mathrm{GL}(de)$, and using the naturality of g with respect to these maps we find that f is a semiring homomorphism. Similarly, we have maps $\lambda^k\colon \mathrm{GL}(d) \to \mathrm{GL}(\binom{d}{k})$ in $\overline{\mathcal{G}}$ and the naturality of g with respect to these maps implies that f commutes with Λ-operations. It is clear that $f(R^+_d(G)) \subseteq \mathrm{Div}^+_d(\mathbb{G})(A)$, so $f \in X_{\mathrm{Ch}}(G)(A)$. We define a map $\rho\colon Y(G) \to X_{\mathrm{Ch}}(G)$ by $\rho(g) = f$. Because $g_{\underline{n}} = g_{n_1} \times \ldots \times g_{n_r}$ we see that ρ is injective.

Now suppose we start with a point $f \in X_{\mathrm{Ch}}(G)(A)$. Let $g_d\colon R(d, G) \to D(d)(A)$ be the restriction of $f\colon R(G) \to \mathrm{Div}(\mathbb{G})(A)$, and put

$$g_{\underline{n}} = g_{n_1} \times \ldots g_{n_r}\colon R(\underline{n}, G) \to D(\underline{n})(A).$$

We need to check that this gives a natural transformation. As $R(\underline{m}, G) = \prod_i R(m_i, G)$ and $D(\underline{m})(A) = \prod_i D(m_i)(A)$, it suffices to check naturality for maps $u\colon \underline{n} \to d$ in $\widetilde{\mathcal{N}}$, or equivalently (by Lemma 16.5) for homomorphisms $u\colon U(\underline{n}) \to \mathrm{GL}(d)$. We need to show that the left hand square in the following diagram commutes:

$$\begin{tikzcd}
R(\underline{n}, G) \arrow[r, "u_*"] \arrow[d, "g_{\underline{n}}"'] & R^+_d(G) \arrow[r, tail] \arrow[d, "g_d"'] & R(G) \arrow[d, "f"] \\
D(\underline{n})(A) \arrow[r, "u_*"'] & \mathrm{Div}^+_d(\mathbb{G}) \arrow[r, tail] & \mathrm{Div}(\mathbb{G})(A).
\end{tikzcd}$$

The right hand square commutes and the two right hand horizontal maps are injective so it suffices to show that the two composite maps $R(\underline{n}, G) \to \mathrm{Div}(\mathbb{G})(A)$ are the same. We call these two maps $\alpha(u)$ and $\beta(u)$. Let F be the set of all functions from $R(\underline{n}, G)$ to $\mathrm{Div}(\mathbb{G})(A)$, thought of as a ring with pointwise operations. It is formal to check that $\alpha(u + v) = \alpha(u) + \alpha(v)$ and $\alpha(uv) = \alpha(u)\alpha(v)$, so α is a homomorphism of semirings from $R^+(U(\underline{n}))$ to $\mathrm{Div}(\mathbb{G})(A)$. It can thus be extended to a ring map $R(U(\underline{n})) \to \mathrm{Div}(\mathbb{G})(A)$, and the same applies to β. It is well-known that $R(U(\underline{n})) = \bigotimes_i R(U(n_i))$ so it suffices to check that $\alpha = \beta$ on $R(U(n_i))$ for all i. This reduces us to the case where $\underline{n} = (e)$ say. It is also well-known that $R(U(e)) = \mathbb{Z}[\lambda^1, \ldots, \lambda^e][(\lambda^e)^{-1}]$, so it suffices to check that $\alpha(\lambda^j) = \beta(\lambda^j)$, which is true because f is a homomorphism of Λ-rings.

This shows that $g \in Y(G)(A)$, and clearly $\rho(g) = f$. Thus ρ is surjective and hence an isomorphism. □

17. A result on restrictions of characters

Theorem 17.1. *Let G be a finite group with a normal subgroup N such that $|N|$ is coprime to $|G/N|$. Then the restriction map $R^+(G) \to R^+(N)^G$ is surjective.*

The proof will follow after some preliminary results.

Lemma 17.2. *Let H be a group, and let W, X, Y be H-sets, with equivariant maps $W \xrightarrow{f} X \xleftarrow{q} Y$. Then there is an equivariant map $\tilde{f}\colon W \to Y$ with $q\tilde{f} = f$ iff for each $w \in W$ there exists $y \in Y$ with $q(y) = f(w)$ and $\mathrm{stab}_H(y) \geq \mathrm{stab}_H(w)$.*

Proof. Write W as a disjoint union of orbits. □

Lemma 17.3. *Let G and N be as above, and let $\rho\colon N \to GL(V)$ be an irreducible representation of N whose character is stable under G. Then there is a homomorphism $\sigma\colon G \to GL(V)$ extending ρ.*

Proof. Suppose $g \in G$, and define $\rho^g\colon N \to GL(V)$ by $\rho^g(x) = \rho(gxg^{-1})$. By hypothesis, this has the same character as ρ, so there exists an intertwining operator $\theta\colon V \to V$ such that $\rho^g(x) = \theta^{-1}\rho(x)\theta$ for all $x \in N$. As V is an irreducible representation of N we see that $\mathrm{Aut}_N(V) = \mathbb{C}$ and thus θ is unique up to multiplication by a scalar matrix. We can thus define a map $\phi\colon G \to PGL(V)$ by $\phi(g) = [\theta]$; this is a homomorphism making the following diagram commute.

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "\rho"'] & G \arrow[d, "\phi"] \\ GL(V) \arrow[r, two heads, "\pi"'] & PGL(V). \end{tikzcd}$$

Put $n = |N|$ and $d = \dim_{\mathbb{C}}(V)$. As V is irreducible we know that d divides n. Put $Y = \{\alpha \in GL(V) \mid \det(\alpha)^n = 1\}$, and note that $\pi\colon Y \to PGL(V)$ is surjective and $\rho(N) \leq Y$. Let N^2 act on G by $(x, y).g = xgy^{-1}$ and on $GL(V)$ by $(x, y).\alpha = \rho(x)\alpha\rho(y)^{-1}$.

We claim that there is an N^2-equivariant map $\zeta\colon G \to Y$ such that $\pi\zeta = \phi$ and $\zeta = \rho$ on N. Clearly $G = N \amalg (G \setminus N)$ as N^2-sets and $\rho\colon N \to Y$ is N^2-equivariant, so it suffices to define ζ on $G \setminus N$. Fix $g \in G \setminus N$, and choose θ as before. After multiplying by a suitable scalar, we may assume that $\det(\theta) = 1$ so $\theta \in Y$. By Lemma 17.2, it will suffice to show that $\mathrm{stab}_{N^2}(g) \leq \mathrm{stab}_{N^2}(\theta)$. Suppose that (x, y) stabilises g, so $xgy^{-1} = g$, so $y = g^{-1}xg$. By the definition of θ we have $\rho(y) = \theta^{-1}\rho(x)\theta$, or in other words $\rho(x)\theta\rho(y)^{-1} = \theta$, so (x, y) stabilises θ, as required.

Now define $\xi\colon G^2 \to Y$ by $\xi(g, h) = \zeta(h)\zeta(gh)^{-1}\zeta(g)$. Clearly $\pi\xi(g, h) = 1$, and the kernel of $\pi\colon Y \to GL(V)$ is the group C_{nd} of nd'th roots of unity, so we can regard ξ as a map $G^2 \to C_{nd}$. As ζ is equivariant, it is easy to check that $\xi(xg, hy) = \xi(g, h)$ for $x, y \in N$, so we get an induced map $\overline{\xi}\colon (G/N)^2 \to C_{nd}$. One also sees directly that for $g, h, k \in G/N$ we have

$$\overline{\xi}(h, k)\overline{\xi}(gh, k)^{-1}\overline{\xi}(g, hk)\overline{\xi}(g, h)^{-1} = 1,$$

so $\overline{\xi}$ is a 2-cocycle. On the other hand nd divides n^2 and thus is coprime to $|G/N|$, so we have $H^2(G/N; C_{nd}) = 0$. We can thus choose a function $\omega\colon G/N \to C_{nd}$ such that $\xi(g, h) = \omega(h)\omega(gh)^{-1}\omega(g)$ for all $g, h \in G$. By putting $g = h = 1$ we see that $\omega(1) = 1$ and thus $\omega(x) = 1$ for $x \in N$. We define $\sigma(g) = \omega(g)^{-1}\zeta(g)$; this clearly gives a homomorphism $G \to GL(V)$ with $\sigma|_N = \rho$, as required. □

Proof of Theorem 17.1. For each irreducible representation ρ of N, let ρ' denote the sum of the inequivalent G-conjugates of ρ. Any G-invariant character is a direct sum of copies of the characters of the representations ρ', so it suffices to show that ρ' extends to a representation of G. Let H be the stabiliser of χ_ρ, so $N \leq H \leq G$. Lemma 17.3 implies that ρ can be extended to a representation σ of H, and one sees from the Mackey formula that $\mathrm{res}^G_N \mathrm{ind}^G_H(\sigma) \simeq \rho'$, so $\mathrm{ind}^G_H(\sigma)$ is the required extension of ρ'. □

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